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Building a better racetrack

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Abstract: We find IIb compactifications on Calabi-Yau orientifolds in which all Kähler moduli are stabilized, along lines suggested by Kachru, Kallosh, Linde and Trivedi.

Keywords: F-Theory, Superstring Vacua.

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Contents

1. Introduction

The problem of stabilizing moduli in superstring compactification has seen much recent progress. A benchmark in this progress is the work of Kachru, Kallosh, Linde and Trivedi [1], which builds on many works on Calabi-Yau compactification, flux compactification and other aspects of string compactification to propose a way to construct $\mathcal{N} = 1$ and non-supersymmetric vacua with all moduli stabilized in a controlled regime. In this paper, we look for specific models of the type they suggest, announce examples which we expect will work in all detail, and explain various senses in which these examples are less common than one might have supposed.

We begin with a short review of the problem. Soon after the pioneering works on Calabi-Yau compactification of the heterotic string [2], it was realized that a disadvantage of this construction was the presence of moduli of the Ricci-flat metrics and vector bundles which it uses. At least in perturbation theory, these moduli and the dilaton become massless scalar fields with at least gravitational strength couplings, whose presence would be in contradiction with experiment.

One might think that this consideration prefers other compactifications with no moduli. However, another possibility is that the problem is just an artifact of perturbation theory. One can see this by postulating simple effective potentials which could plausibly emerge from non-perturbative effects in string theory, which have isolated minima. Any of these minima would be a vacuum with stabilized moduli.

Some early examples were the models of [3, 4], and the "racetrack" models [5, 6, 7] which use non-perturbative contributions from two gauge theory sectors, with different dependences on the gauge coupling. Many similar constructions have been proposed over the years. Their essential ingredient, more than any particular feature of string theory, is the concept of effective potential, in which effects from various different sources are added into a single function which controls the vacuum structure. Once one believes that this is an accurate picture of the situation, then given a sufficiently rich supply of contributions to the potential, it becomes very plausible that the typical potential will have many isolated minima, so that moduli stabilization is generic. Furthermore, the value of the effective potential at the various minima (the "effective cosmological constant") will be widely distributed among both positive and negative values, possibly including the observed value.

One might ask whether other general features of the problem, such as low energy supersymmetry, change this general expectation. As we have discussed at length elsewhere [8], the standard expression for the potential from $\mathcal{N} = 1$ supergravity, supports this point of view. By postulating a superpotential and Kähler potential, one can get a wide range of potentials, which do not obviously favor positive or negative effective cosmological constant.

One important general expectation from string theory is the following. Calabi-Yau compactification, and essentially any "geometric" compactification of string and M-theory, has a "large volume limit" which approaches ten or eleven dimensional Minkowski space. In this limit, the four dimensional effective potential will vanish [9]. Recently more general arguments for this claim were made by Giddings [10]. This makes it somewhat harder to stabilize vacua with zero or positive effective cosmological constant, as one needs the potential to have at least two points of inflection (the desired minimum, and a maximum or more generally a barrier at larger compactification radius). Still, this leads to no obvious problem of principle.

To go beyond these rather general claims, one must get sufficient control over nonperturbative effects to show that such potentials could arise from string/M-theory, and be able to compute them in a wide enough range of examples to make serious predictions. This line of thought developed in the late 1980's, and led to a focus on nonperturbative methods in string theory, a field which took off with the 1989–90 work on matrix models. An equally important early nonperturbative result was the exact solution, including all world-sheet instanton corrections, of $(2,2)$ Calabi-Yau sigma models using mirror symmetry [11].

While when first formulated these techniques appeared to address only low dimensional models, the 1993–94 work of Seiberg and collaborators on nonperturbative supersymmetric gauge theory showed that similar results could be obtained for the effective superpotential in simple four dimensional models. With the 1994–95 realization [12, 13, 14] that duality and branes were key nonperturbative concepts, tremendous advances were made, leading to a fairly good picture of compactifications with at least 8 supercharges, and some success in describing compactifications with less or no supersymmetry.

Studying such compactifications and mapping out the possibilities requires both the techniques to do Calabi-Yau, F theory, G_2 compactification etc. and study the physics of each, and "classification" and "mapping" results, such as the classification of of Calabi-Yau's which are toric hypersurfaces, and the detailed study of their moduli spaces, for example [15]. These should both provide specific examples of these general recipes and some idea of which ones might be relevant for describing the real world. We will call upon a number of these results below [16]–[21].

The subject of compactification with four or fewer supercharges remains somewhat controversial. Most works on this subject freely use field theoretic concepts such as Kaluza-Klein reduction and the effective potential, with the main string/M-theory inputs being new light states and new corrections to the effective lagrangian. At present, there is no real evidence that this is wrong.

On the other hand, it is true (by definition) that a vacuum with stabilized moduli is not connected to the limit of arbitrarily weak coupling and arbitrarily large compactification volume through time-independent solutions. One must pass either through the larger configuration space ("go off-shell" in an old-fashioned language) or consider time-dependent solutions to do this, and our lack of understanding of these subjects in string/M-theory makes it conceivable that what look like valid solutions from an effective field theory point of view in fact are not valid solutions of string/M-theory. This possibility has been put forward most forcefully by Banks and collaborators [22, 23]. While we do not find their present arguments convincing, the point is definitely not settled and deserves attention.

Another issue along these lines is the question of whether there is some obstacle within string/M-theory to realizing compactifications with positive vacuum energy. Over the mid-90's, convincing observational evidence was found for an accelerated expansion of the universe, which is most simply explained by the hypothesis that there is a non-zero positive

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"dark energy" of energy density about 0.7 times the critical value. While there are many possibilities for what this is, the simplest and probably easiest to reproduce from string theory is a cosmological constant, which would appear to cause the long term evolution of the universe to asymptote to a de Sitter geometry. From the point of view of field theory and the effective potential, there is no problem with getting positive vacuum energy. On the other hand, no such string vacuum is known, and theoretical arguments were even advanced that these should not exist [22, 24, 25].

Meanwhile, progress in our understanding of string/M-theory has led to steady progress in the study of moduli stabilization. An important recent work in this direction is that of Kachru, Kallosh, Linde and Trivedi [1], who argue that metastable de Sitter vacua with all moduli stabilized can be constructed in type IIb string theory, at least in an effective potential framework.

The KKLT work has two aspects. In the first, they propose a recipe for such a construction, which we will discuss in detail shortly. In the second, they sidestep the claim that de Sitter space could not be realized in string theory, by suggesting that our universe need not asymptote to de Sitter space. Rather, our vacuum might be metastable, in a positive local minimum of the effective potential. As we dicussed, since the potential goes to zero at large compactification volume, any such vacuum is potentially metastable to decay to this decompactification limit. Using results of Coleman and de Luccia, Hawking and Moss, and others on tunneling in semiclassical quantum gravity, KKLT argued that for almost any reasonable potential, one would find an extremely small but non-zero decay rate, slow enough to pose no cosmological constraint, but fast enough to evade the paradoxes which had been suggested might forbid eternal de Sitter space as a vacuum. These ideas are reviewed in [26] (see also the recent [27]).

This leaves the more technical problem of showing that such metastable de Sitter vacua actually do exist in string theory compactification. KKLT suggested a recipe by which this could be done, following previous work of many authors on moduli stabilization by fluxes (a very incomplete list includes [28]–[39]) and especially Giddings, Kachru and Polchinski [40] who developed a class of IIb flux compactifications which stabilize the dilaton and complex structure moduli. Some advantages of this construction are that it allows fixing the dilaton to weak coupling by making an appropriate choice of flux, and that cancellations can bring the flux contribution to the potential below the string scale. Most importantly, it is based on the relatively well understood theory of Calabi-Yau moduli spaces, so explicit computations can be done.

Having this construction in hand, the remaining problem is to stabilize the Kähler moduli and obtain a small positive cosmological constant. Granting the effective potential framework, one proceeds as in the early works on moduli stabilization, to look for nonperturbative contributions to the superpotential which could stabilize the remaining moduli. As it happens, in IIb compactification nonperturbative effects from gauge theories on branes, and from D-instantons, all depend on Kähler moduli, and thus one would expect that generically this stabilization would not be a problem — once one had computed these constributions to the effective potential sufficiently well, they would stabilize all moduli.

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Finally, to obtain a positive cosmological constant, one has several choices. Conceptually the most straightforward is to look for supersymmetry breaking vacua of the previously computed effective potential, which again on grounds of genericity should exist. Alternatively, one can look for supersymmetric AdS vacua which stabilize all moduli, and then add other supersymmetry breaking effects which lift the vacuum energy. This idea has the advantage that the resulting potential can naturally have the barrier required for metastability of a de Sitter vacuum. KKLT suggested to add anti D3-branes, and one can imagine many other possibilities such as D terms coming from brane sectors [41, 42, 43].

The upshot is that such effects, if they exist, could well combine to stabilize all moduli at a metastable de Sitter minimum. There are two obvious gaps in this argument. The deeper one is the question of whether the effective potential analysis is valid. Perhaps the best way to address this is to consider a specific example, and try to justify all of its ingredients in ten dimensional string theory.

To do this, one must address the first gap, which is that no concrete model of this type has yet been put forward. In this work, we will fill this gap, by examining models of the general class discussed by KKLT, to see if this idea can actually be implemented in string theory. We will need to call upon many of the ideas discussed above, especially the classification of Calabi-Yau three-folds and four-folds, and the analysis of nonperturbative effects in F theory by Grassi [16].

While we will exhibit models in which nonperturbative effects lift all Kähler moduli, we will also argue that, at least if we rely on instanton effects, such models are *not* generic. There is a fairly simple reason to expect this. In the language of orientifold compactification with branes, it is that these world-volume gauge theories usually have too much matter to generate superpotentials, as seen in works such as [44], and as we will see in a large class of models below. This is because the cycles on which the branes wrap can be deformed, or because the branes carry bundles with moduli. In either case, the gauge theory has massless adjoint matter, which eliminates the superpotential. The simplest case is the 0-cycle or point, which clearly always has moduli. Of the two-cycles, only S^2's can be rigid; higher genus Riemann surfaces in Calabi-Yau almost always come with moduli. Bundles on surfaces (and the entire CY) normally have moduli as well.

On the other hand, in a sizable minority of models, the instanton generated superpotentials are sufficiently generic to stabilize all Kähler moduli. The examples that we can check systematically are toric fourfolds X elliptically fibered over a three dimensional base B which is Fano, and $\mathbb{P}^1$-bundles over a toric surface. Much of what we say should hold more generally.

There are no very simple models in this class. In particular, one can argue that no model with one Kähler modulus can work.[1]

Of the models which work, we have considered three of the simpler ones in some detail. The fourfolds are elliptic fibrations over the Fano threefolds $\mathcal{F}_{11}$ and $\mathcal{F}_{18}$ [20, 21, 46]. Their Euler characteristics are $\chi = 16848$ and $\chi = 13248$ respectively. Some of our considerations

[1]This conclusion was reached independently by D. Robbins and S. Sethi [45]. Also, relevant work of V. Balasubramanian and P. Berglund is mentioned in [26].

will require that we work with their orientifold limits. These are Calabi-Yau threefolds that can be realized as hypersurfaces in toric varieties and their Hodge numbers are $h^{1,1} = 3, h^{2,1} = 111$ and $h^{1,1} = 5, h^{2,1} = 89$ respectively. Finally, we consider an elliptic fibered CY over $\mathbb{P}^2$, a much studied CY [47] with $h^{1,1} = 2$ and $h^{2,1} = 272$.

2. Review of KKLT construction

We start with IIb superstring theory compactified on a Calabi-Yau orientifold, specified by a choice of CY threefold Z and a holomorphic involution $\hat{\Omega}$ on Z whose fixed locus consists of points and surfaces (O3 and O7 planes). For example, we might take Z to be the hypersurface in $\mathbb{CP}^4$ defined by a quintic polynomial $f(z) = 0$, and $\hat{\Omega}$ to be the map $z_1 \to -z_1$, $z_i \to z_i$ for $i > 1$.

We then introduce D-branes in such a way as to cancel the RR tadpoles produced by the orientifold planes, as in [48]. For a recent discussion of this problem, see [49].

The number of complex structure moduli of Z is $h^{2,1}(Z)$, and the number of complexified Kähler moduli is $h^{1,1}(Z)$. We are going to refer to these as "shape" and "size" moduli respectively, mostly because the words "complex" and "Kähler" have so many different roles in the discussion that the usual terminology is cumbersome (one gets tired of saying "Kähler metric on the Kähler moduli space," etc.)

A more general starting point would be F theory [50]. We will use this language to construct models below, but rapidly move to the IIb orientifold limit in our examples, since there are many unanswered physical questions in either picture, which we will need to appeal to the underlying definitions in string theory to resolve.

The basic relation is as follows. An F theory compactification is defined by a choice of Calabi-Yau fourfold X with an elliptic fibration structure over a threefold B,

$$\begin{tikzcd} T^2 \arrow[r] & X \arrow[d, "\pi"] \\ & B. \end{tikzcd} \tag{2.1}$$

Physically, we compactify IIb theory on B, introducing 7-branes at the singularities of the fibration π, so that the resulting dilaton-axion field at a point $p \in B$ corresponds to the complex structure modulus τ of the fiber $\pi^{-1}(p)$. The orientifold limit [51] is then the special case in which all of these singularities are D_4 singularities (an O7-plane and four coincident D7's). In this case, Z is a double cover of B branched at the singularities, and $\hat{\Omega}$ exchanges the two sheets. Let us denote the part of the cohomology of Z even or odd under $\hat{\Omega}$ with subscripts $\pm$.

Classically and in the absence of fluxes, this compactification has a moduli space of $\mathcal{N} = 1$ supersymmetric vacua, parametrized by $h^{2,1}_-$ shape moduli, the dilaton-axion, $h^{1,1}_+$ size moduli complexified by the RR 4-form potential C_4, and $h^{1,1}_-$ moduli from the 2-form potentials B_2 and C_2 [44, 52].

We will avoid having to discuss the 2-form moduli by restricting attention to models with $h^{1,1}_- = 0$. More generally, while we do not know a mechanism which would stabilize them at large volume, they might be stabilized by world-sheet and D1-instantons, or by couplings to brane world-volumes.

In general, there are also open string moduli corresponding to positions of D7-branes and D3-branes, and moduli of bundles on D7-branes. We will ignore these through most of the discussion and return to them in the conclusions. One excuse for this is that these moduli spaces are all expected to be compact. This is intuitively clear for positions of D3 branes, since their moduli space is Z itself, which is compact. This should also be true for moduli spaces of bundles except for small instanton limits, but these just correspond to other brane configurations. Thus, there is no analog of the decompactification or runaway problems for these moduli, and generic corrections to the potential should fix them.

2.1 Fixing moduli using fluxes

We begin by trying to fix the shape moduli and the dilaton-axion. In IIb language, this is done by turning on the NS and RR three-form field strengths, $H^{(3)}$ and $F^{(3)}$ respectively. As discussed in many references (e.g. [40, 53]) the equations of motion will force these to be harmonic forms, so they are determined by their cohomology classes in $H^3(Z,\mathbb{R})$. There is a Dirac-type quantization condition which normally forces these classes be integer quantized in units of the string scale. Setting this unit to one, they live in $H^3(Z,\mathbb{Z})$.

The choice of flux is constrained by the tadpole condition on the RR four-form potential,

$$L \equiv \frac{1}{2}\int G_4 \wedge G_4 = \frac{1}{2} N_R^\alpha \, \eta_{\alpha\beta} \, N_{NS}^\beta = \frac{\chi(X)}{24} - N_{D3} \,, \tag{2.2}$$

where N_{D3} is the number of D3-branes minus the number of anti-D3 branes.

The ten-dimensional IIb supergravity analysis of unbroken $\mathcal{N}=1$ supersymmetry in this context is done in [34]; the most important condition is that

$$G^{(3)} = F^{(3)} - \tau H^{(3)}$$

must be imaginary self-dual, $G = i * G$. One can restate this as the condition that

$$G^{(3)} \in H^{(0,3)}(Z,\mathbb{C}) \oplus H^{(2,1)}(Z,\mathbb{C}) \,.$$

Setting the remaining parts of the cohomology to zero amounts to $h^{2,1}+1$ conditions, which is precisely enough to fix all shape moduli and the dilaton. Thus, we might expect generic choices of flux to do this.

There is an additional condition for this to be true with four noncompact Minkowski dimensions: the $H^{(0,3)}$ component of $G^{(3)}$ must be zero. This is one more condition than the number of moduli, so this is non-generic. Otherwise, we get supersymmetric AdS vacua.

A nice physical summary of these results, which will be essential later on, is that these conditions on $G^{(3)}$ follow from solving the supersymmetry conditions in an $\mathcal{N}=1$ effective supergravity theory. Its configuration space is the combined Calabi-Yau and dilaton-axion moduli space; in other words the product of the shape moduli space $\mathcal{M}_c(Z)$, the size moduli space $\mathcal{M}_K(Z)$, and the upper half plane $\mathcal{H}$, with the Kähler potential

$$\mathcal{K} = -\log \operatorname{Im} \tau - \log \int_Z \Omega \wedge \bar{\Omega} - 2\log V \tag{2.3}$$

where τ is the dilaton-axion, V is the volume of the Calabi-Yau as a function of the size moduli (more on this below), and Ω is the holomorphic three-form on Z. This is just the Kähler potential of the compactification with zero flux and $\mathcal{N} = 2$ supersymmetry, computed in the $\alpha', g_s \to 0$ limit.

As superpotential, we take the Gukov-Vafa-Witten superpotential

$$W_{GVW} = \int \Omega \wedge G^{(3)} \,. \tag{2.4}$$

One can see [40, 53] that the conditions

$$0 = D_i W = \partial_i W + (\partial_i \mathcal{K}) W \tag{2.5}$$

for the shape moduli set the $H^{(1,2)}$ part of $G^{(3)}$ to zero, and for the dilaton-axion sets the $H^{(3,0)}$ part to zero.

For typical values of the flux, this will fix τ and the shape moduli z at a mass scale $m_0 \sim \alpha'/\sqrt{V}$, where V is the volume of Z. The typical value of $e^{\mathcal{K}}|W|^2$ at the minimum is 1 in string units, but this can be made smaller by tuning the fluxes. One expects its smallest value to be $\sim 1/\mathcal{N}_{\text{vac}}$ where $\mathcal{N}_{\text{vac}}$ is the number of flux vacua on the complex structure moduli space, as we discuss below and in [8].

We will discuss the technicalities of finding choices of flux which stabilize these moduli in a desired region of moduli space in section 4. It will turn out that this is often, but not always possible.

2.2 Fixing size moduli

Since (2.4) is independent of the size moduli, the conditions $D_i W_{GVW} = 0$ for the size moduli amount to either $\partial_i \mathcal{K} = 0$ for all size moduli, or else $W = 0$. It is easy to see (as we discuss below) that in the large volume limit, one cannot solve $\partial_i \mathcal{K} = 0$ for all moduli. One can find Minkowski supersymmetric solutions with $W_{GVW} = 0$. One can also find Minkowski but non-supersymmetric vacua with $W_{GVW} \neq 0$, because of the no-scale structure of (2.3).

In either case, while we might find other effects which lead to a positive vacuum energy, this energy will typically decrease with increasing compactification volume, and the resulting solutions will be unstable. We need a more complicated effective potential to fix this problem, which might be obtained by incorporating stringy and quantum corrections to W and $\mathcal{K}$.

Unfortunately doing this remains a hard problem, both to get results in examples, and even to define what one means by the nonperturbative effective potential in general. In particular, there are essentially no results on nonperturbative corrections to the Kähler potential.

To address this, one can seek examples where the instanton expansion appears to be valid, so that the leading corrections will dominate. This was the primary goal in KKLT's discussion and we will address it below. However we believe that there is no question of principle which requires restricting attention to these examples; rather we do this because of our limited technical control over the theory at present.

KKLT suggested to stabilize the size moduli using quantum corrections to the superpotential, as they are known to be present and depend on the size moduli in many examples, and are controlled at large volume and weak coupling.

For example, one can consider a case in which D7-branes wrapped on a cycle Σ have pure SU(N) Yang-Mills theory as their world-volume theory. The gauge coupling in this theory is $1/g^2 = V(\Sigma)$ where $V(\Sigma)$ is the volume of the cycle Σ in string units, and thus nonperturbative effects in this theory lead to a superpotential

$$W_{\mathrm{eff}} = B\, e^{-V(\Sigma)/N} \tag{2.6}$$

where B is a (presumably order 1) quantity determined by threshold effects, etc. and which may also depend on size and shape moduli.

If we consider a vacuum in which the flux contribution to the superpotential (2.4) takes a value W_0 independent of size moduli, and grant that this is the leading correction, the supersymmetry condition for the size modulus becomes (let $\rho = iV$),

$$0 = D_\rho W = -\frac{B}{N} e^{i\rho/N} - \frac{3}{\operatorname{Im}\rho}\left(W_0 + B e^{i\rho/N}\right). \tag{2.7}$$

For small W_0, this will normally have a unique solution at large imaginary ρ, roughly given by

$$\rho \sim -i \log \frac{3NW_0}{B}.$$

To see this without detailed calculation, note that the problem of solving $D_\rho W = 0$ with $W \neq 0$ is the same as the problem of finding critical points of the function $f = e^{\mathcal{K}}|W|^2$. This function is positive and goes to zero at very large $\operatorname{Im}\rho$. If we contrive W to have a zero at large $\operatorname{Im}\rho$, then it is easy to see that as we increase $\operatorname{Im}\rho$ from this value at fixed $\operatorname{Re}\rho$, we are always at a critical point in $\operatorname{Re}\rho$. Furthermore, as we increase $\operatorname{Im}\rho$, f starts out increasing, so for it to later decrease it must pass through a critical point in $\operatorname{Im}\rho$.

This solution will have $W \sim W_0$ and thus we will end up with a supersymmetric AdS solution with small negative cosmological constant, controlled by the previous step where we chose fluxes to get small W_0.

Another source of similar non-perturbative contributions to the superpotential is D3-brane instantons wrapped on surfaces in Z [54]. Their dependence on the size moduli is the same and thus this level of the discussion works the same way. As we discuss below, in the case that D7-branes wrap a cycle Σ, these are equivalent to gauge theory instantons, but are present more generally. Thus in the detailed discussion below, we consider the non-perturbative corrections as generated by D3-brane instantons.

Some further general observations on this problem appear in [55].

2.3 Breaking supersymmetry

We will have enough trouble trying to realize the first two steps, but given this, to go on to the third step we would need some approximate expression for the warp factor on Z, so that we could argue that the increase in vacuum energy produced by adding an anti

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D3-brane could also be made small. The simplest way to imagine this working is to fix the shape moduli near a conifold singularity, and appeal to the Klebanov-Strassler solution [56] as an approximate description of the metric in this region.

While it seems to us that this should generally work, of course there is room for more subtleties, as we will briefly discuss below.

3. Details of size moduli stabilization

We seek the conditions on Z which will lead to a superpotential which stabilize all size moduli significantly above the string scale, so that α' corrections can be ignored. It will emerge from considerations below that there are no models with $h^{1,1} = 1$, and thus we consider the multi-moduli case from the start. We will usually set $\ell_s = 2\pi\sqrt{\alpha'}$ to 1 in the following.

Let D_i be a basis of divisors on Z (essentially, these are classes in $H_4(Z, \mathbb{Z})$). The main data we will need about Z are the triple intersection numbers of the divisor basis, $D_{ijk} = D_i \cdot D_j \cdot D_k$. Using Poincaré duality, we also let D_i denote the corresponding class in $H^2(Z, \mathbb{Z})$. For CY's which are hypersurfaces in toric varieties, there are efficient methods for computing these intersection numbers and the other data we are about to discuss. We discuss some relevant examples in section 5 and appendix B.

We define the size moduli by writing the Kähler class J of Z as

$$J = \sum_i t^i D_i \,, \tag{3.1}$$

defining a set of real coordinates t^i on the space of Kähler classes.

The classes which actually correspond to Kähler metrics are those lying in the Kähler cone, defined by

$$0 < \int_C J \tag{3.2}$$

for all holomorphic curves C. The classes of such curves are called "effective classes" and form a cone, the Mori cone.

In fact, to be able to ignore α' corrections, we want all such areas of curves to be at least $\mathcal{O}(1)$ in string units.

The natural holomorphic coordinates which appear in IIb orientifold compactification are not the t_i, but are instead the complexified volumes of divisors, which we denote τ_i,

$$\tau_i = \int_{D_i} \frac{1}{2} J \wedge J - i\, C_4 \,. \tag{3.3}$$

We denote their real parts as

$$V_i = \mathrm{Re}\, \tau_i \,.$$

The Kähler potential on this configuration space, neglecting α' and g_s corrections, is

$$\mathcal{K}_K = -2 \ln V(\tau, \bar{\tau}) \tag{3.4}$$

where V is the volume of Z, defined by

$$V = \frac{1}{6} J^3 = \frac{1}{6} D_{ijk} t^i t^j t^k \,. \tag{3.5}$$

We can write this (implicitly) as a function of τ_i using

$$\tau_i = \frac{\partial V}{\partial t_i} = \frac{1}{2} D_i J^2 = \frac{1}{2} D_{ijk} t^j t^k \,. \tag{3.6}$$

This change of coodinates is essentially a Legendre transform on V, as discussed in detail in [57].

3.1 Instanton superpotentials

The best studied corrections to the superpotential in F theory are produced by D3-brane instantons wrapped around divisors. These take the form

$$W_{np} = \sum_{\vec{n}} b_{\vec{n}} \, e^{-2\pi \vec{n} \cdot \vec{\tau}} \tag{3.7}$$

where the $b_{\vec{n}}$ are one-loop determinants on the divisors $D_{\vec{n}} = n^i D_i$, depending on the complex structure moduli. At large radius, $2\pi\tau_i \gg 1$ and the mass scale of this contribution to W is exponentially suppressed compared to m_0, so to a good approximation we can consider the complex structure moduli to be fixed by the tree level flux superpotential W_0, and use $W = W_0 + W_{np}$ with constant W_0 and $b_{\vec{n}}$ as effective superpotential for the Kahler moduli.

The most important thing to know about the determinants $b_{\vec{n}}$ is whether they are non-vanishing. This was studied in [54] by using the relation between F theory and M-theory compactification on the four-fold. In M-theory, such a correction comes from a fivebrane instanton wrapped on the lift $\mathcal{V}_{\vec{n}}$ of the divisor to the fourfold. The F theory limit is the limit in which the area of the fiber goes to zero, and a divisor $\mathcal{V}$ will contribute in this limit if it is vertical, meaning that $\pi(\mathcal{V})$ is a proper subset of B.

In this formulation, $b_{\vec{n}}$ includes the determinant of a Dirac operator on $\mathcal{V}$, which can have zero modes. In fact the zero modes turn out to be equivalent to holomorphic p-forms on $\mathcal{V}_{\vec{n}}$ tensored with a three-dimensional spinor. Let the number of such p-forms be $h^{0,p}$.

An instanton which contributes to W_{np} must have two fermion zero modes (corresponding to the two supersymmetries broken by the instanton). Since $h^{0,0} = 1$, this will be true if $h^{0,j} = 0$ for all $j = 1, 2, 3$. On the other hand, it is possible for other brane couplings (in particular, couplings to the NS and RR fluxes we used to stabilize other moduli) to lift zero modes in pairs. Thus it is possible that $b_{\vec{n}} \neq 0$ more generally, but in any case the necessary criterion for this is that $\mathcal{V}_{\vec{n}}$ is a divisor of arithmetic genus

$$\chi(\mathcal{O}_{\mathcal{V}}) \equiv \sum_j (-1)^j h^{0,j} = 1 \,.$$

In this more general case, it is also possible for variations in complex structure to bring down or lift zero modes in pairs, so $b_{\vec{n}}$ could have zeroes at special points in complex structure moduli space, possibly depending on the choice of flux.

Unfortunately not much more is known about the determinants $b_{\vec{n}}$. Note however that they must be sections of the same line bundle as the holomorphic 4-form Ω_4 on X; otherwise the combined superpotential wouldn't make sense. Let us fix the overall scale of Ω_4 (and thus $b_{\vec{n}}$) by putting $\int_X \Omega_4 \wedge \bar{\Omega}_4 = 1$ at the given point (τ, z). In this natural normalization, it is reasonable to assume the $b_{\vec{n}}$ are generically not much smaller or bigger than 1.

The upshot is that divisors whose lift to X have $h^{0,j} = 0$ for all $j = 1, 2, 3$ will contribute to the superpotential, whether or not flux is turned on and regardless of the values of the other moduli. The most general class of divisors which can contribute are the divisors of arithmetic genus 1; these contributions might vanish for special values of flux and moduli.[2]

3.2 Gauge theory superpotentials

Superpotentials produced by non-perturbative effects in gauge theories living on D7-branes can also be understood as coming from divisors of arithmetic genus 1.

Roughly, if one wraps a D7 about a divisor Σ, the zero size limit of the Yang-Mills instanton in the resulting world-volume gauge theory is just a D3-instanton wrapped on Σ. A detailed discussion is somewhat more complicated. For example, in pure $\mathrm{SU}(N_c)$ super Yang-Mills theory, the instanton has $2N_c$ fermion zero modes, which for the corresponding D3-instanton arise from 3–7 open strings. Thus it does not contribute directly to the superpotential. Rather, its effects force gaugino condensation which leads to a vacuum energy which can be reproduced by a superpotential. This accounts for the $1/N$ in the exponent of (2.6).

In F-theory, nonabelian gauge symmetries arise from singularities of the elliptic fibration. For example, an $\mathbf{A}_{N-1}$ singularity fibered over a surface S in the base corresponds to an $\mathrm{SU}(N)$ gauge theory living on S. If S has $h^{0,1} = h^{0,2} = 0$ and the singularity does not change over S, there will be no additional matter and we are in the pure $\mathrm{SU}(N)$ case just discussed. In [61], it was shown that by compactifying on a circle and going to the dual M-theory picture, the three dimensional nonperturbative superpotential can be computed as coming from M5 brane instantons wrapping the exceptional divisors obtained by resolving the $\mathbf{A}_{N-1}$ singularity. Again, these divisors are of arithmetic genus 1. The correct four dimensional result is then obtained by extremizing the three dimensional superpotential and taking the decompactification limit.

A similar geometrical analysis can be done when in addition fundamental matter is present [62]. The situation is more subtle here, but the upshot is that the gauge theory superpotentials can again be derived from M5 brane instantons wrapped on divisors of arithmetic genus 1.

[2]One might worry about possible modifications to this analysis if $\mathcal{V}$ is spin_c and not spin, for reasons discussed in [58]. These can modify the coefficient b and might even introduce dilaton dependence [59], but do not affect the statements made here or our later conclusions. We thank G. Moore and S. Sethi for communications on this point.

We also note that the analysis in [54] used a U(1) symmetry of the normal bundle to the divisor. If this were broken by the fluxes, the arithmetic genus 1 condition might be relaxed [60].

Thus, in the known examples, a necessary condition to generate a nonperturbative superpotential is the presence of a divisor of arithmetic genus 1 in the fourfold.[3]

A more intuitive but only partial argument for the role of the arithmetic genus in the brane gauge theory is that the cohomology groups $H^{0,p}(\mathcal{V})$ which entered our previous discussion also control the matter content on the D7-brane gauge theories. In particular, non-zero $h^{0,1}$ and $h^{0,3}$ will lead to massless adjoint matter, which tends to eliminate the superpotential (as in $\mathcal{N}=2$ gauge theory). This conclusion is not strict as gauge theory with a non-trivial world-volume superpotential for the adjoint matter (say $\mathrm{Tr}\,\phi^n$) can generate a superpotential, but this possibility comes along with the possibility of deforming the superpotential to give mass to the adjoint matter and thus is accounted for by the possibility we discussed earlier of lifting the zero modes in pairs (this can be made more precise by relating the superpotential to obstruction theory and $h^{0,2}$ as in the D-brane literature [63]).

3.3 Complete sets of divisors

Divisors of arithmetic genus 1 tend to be rare [54, 16]. Furthermore, in order to stabilize the compactification at strictly positive radii, a sufficient number of distinct divisors must contribute. This can be seen as follows.

We are looking for solutions to $D_{\tau_i}W = \partial_{\tau_i}W + (\partial_{\tau_i}\mathcal{K})W = 0$. Now

$$d\mathcal{K}_K = -\frac{1}{V}J^2 dJ = -\frac{1}{2V}JdJ^2 = -\frac{t^i dV_i}{V}\,, \tag{3.8}$$

hence $\partial_{\tau_i}\mathcal{K} = -\frac{t^i}{2V}$, and

$$0 = D_{\tau_i}W = \sum_{\vec{n}} b_{\vec{n}}\, e^{-2\pi\vec{n}\cdot\vec{\tau}}(-n_i)\; - \;\frac{t^i}{4\pi V}W\,. \tag{3.9}$$

By multiplying with D_i and summing, it follows that

$$\sum_{\vec{n}} b_{\vec{n}}\, e^{-2\pi\vec{n}\cdot\vec{\tau}} D_{\vec{n}} \; + \frac{W}{4\pi V}J = 0\,. \tag{3.10}$$

We want vacua with $W \neq 0$. This is generic, and if the contributing divisors are linearly independent, as is often the case in examples, it follows directly from this equation.

In this case, the Kähler class J is a linear combination of the contributing divisors. Thus, a necessary condition for a solution with positive radii to exist is that a linear combination

$$\mathcal{R} = r_{\vec{n}} D_{\vec{n}}$$

of the contributing divisors exists such that $\mathcal{R}$ lies within the Kähler cone (3.2). We will call such a set of divisors "complete."

[3]Actually, in [54], a more general possibility is raised: divisors of arithmetic genus $\chi(D) > 1$ (but presumably never $\chi \leq 0$) might contribute to the superpotential, through strong infrared dynamics or "fractional instanton" effects. We thank S. Kachru for reminding us of this point.

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Note that a set of divisors cannot be complete if there exists a curve C which does not intersect any divisor in that set, because then all $\mathcal{R}$ will lie outside the Kähler cone, or at best on its boundary. On the other hand a full basis of divisors is automatically complete. Since it is easy to verify if a set of divisors is a basis, we will focus mainly on examples in which a full basis in $H_4(Z)$ of divisors contributes to the superpotential.

3.4 Parameters

At large volume, the exponentials in (3.10) are very small, so for a solution to exist, W and therefore W_0 has to be very small, of the order of the dominant exponentials $e^{-2\pi\vec{n}\cdot\vec{\tau}}$. Again because J in (3.10) has to lie well inside the Kähler cone, the set of contributing divisors $D_{\vec{n}}$ which have $e^{-2\pi\vec{n}\cdot\vec{\tau}} \sim W_0$ has to be complete in the sense introduced above, since the terms for which $e^{-2\pi\vec{n}\cdot\vec{\tau}} \ll W_0$ can effectively be neglected.

For a suitable choice of basis of divisors, the t^i in (3.1) correspond to curve areas, so a rough estimate for the divisor volumes is $V_i \sim kA^2$, where A is the average curve area and k some constant, which increases with the number of Kähler moduli due to the increase in terms in (3.6) (in examples k is at least proportional to the number of Kähler moduli). So an estimate for the average curve size at a critical point of the superpotential is

$$A \sim \left(\frac{\ln |W_0|^{-1}}{2\pi k}\right)^{1/2} . \tag{3.11}$$

We see from this estimate that for many Kähler moduli (so k is large), W_0 has to be exceedingly small to have $A > 1$.

Moreover for many moduli the probability that some curves are significantly smaller than the average increases, making it even more difficult to fix the moduli in a controlled regime. This motivates us to look for models with few Kähler moduli.

3.5 Geometric considerations

There are some general geometric constraints on the existence of divisors of arithmetic genus 1. First, we can assume without loss of generality that there exists a Weierstrass model $\pi_0 : W \to B$ and that $\mu : X \to W$ is the resolution of the Weierstrass model. We recall that in F theory only the vertical divisors of arithmetic genus one contribute to the superpotential. As explained in [16], such divisors fall in two classes: they are either components of the singular fibers or they are of the form $\mathcal{V} = \pi^*(D)$ for D a smooth divisor on B.

To construct elliptic fourfolds which have divisors of the first type, one can take B to be a $\mathbb{P}^1$-bundle over a surface B' and enforce an ADE singularity of the Weierstrass model along B', as in [61]. The exceptional divisors of $\mu : X \to W$ will have arithmetic genus one. In IIb string theory terms, this corresponds to realizing pure Yang-Mills theory with ADE gauge group on 7-branes (D7-branes in the A and D cases, and for the E case as in [64]) and wrapping these branes on B'. The arithmetic genus one condition then corresponds to the condition $h^{0,1}(B') = h^{0,2}(B') = 0$ for the resulting four dimensional gauge theory not to have matter. Because of the association of these divisors to brane gauge theory we refer to them as "gauge-type divisors."

The other class of divisor, pullbacks of smooth divisors on B, can give rise to D3-brane instanton corrections which need not have a gauge theory interpretation, and so we refer to these as "instanton-type divisors." In [54, 65, 19, 46] it was proposed, based on the study of examples, that such divisors are always "exceptional" in the sense that there exist birational transformations of Calabi-Yau fourfolds which contract these divisors. This was shown to be true whenever B is Fano in [16] . Moreover, when B is Fano or toric, the number of the divisors contributing to the superpotential is finite. This is because they are the exceptional divisors associated to the contraction of one of the generators of the Mori cone, which is polyhedral for B Fano or toric.[4] These divisors have a negative intersection with the corresponding generator of the Mori cone and are thus "non-nef," as first pointed out in [66].

One can now see that there are no models with $h^{1,1} = 1$, because the negative intersection condition in this case reduces to

$$\Sigma \cdot D < 0$$

where $D = \pi(\mathcal{V})$ and Σ is an effective curve.[5] The positivity condition (3.2) then forces the Kähler form to be $J = -tD$ with $t > 0$. The conditions $V = J^3/6 > 0$ then forces $Vol(D) = D \cdot J^2/2 < 0$, so such an instanton correction cannot exist. In words, the "exceptional" nature of these divisors means that in each of the instanton amplitudes (3.7), at least one of the coefficients in the action $\vec{n} \cdot \vec{\tau}$ must be negative (in some basis). Given several Kähler moduli and divisors, this need not be a problem, and we will find in examples that it is not, but it does preclude $h^{1,1} = 1$.

A more mathematical argument for this point is that if arithmetic genus one divisors are indeed always exceptional, then there are no models with $h^{1,1} = 1$, because contracting a divisor will decrease $h^{1,1}$ (from the "Contraction Theorem" cited in [16]), which is impossible in this case.

It is also shown in [16] that for B Fano, divisors of arithmetic genus one in fact have $h^{0,p} = 0$ for $p > 0$, so the necessary condition for an instanton correction is in fact sufficient.

4. Details of flux stabilization

In the next section, we will find that models which will stabilize all Kähler moduli have many complex structure moduli, $n \sim 100$. We will then need to find choices of flux which stabilize the other moduli and lead to small W_0. While one can search for solutions numerically, as done in previous work [36, 67, 68], computation time becomes an important issue. In particular the most straightforward approach of picking arbitrary flux vectors, trying to find solutions to $DW_0 = 0$, and hoping the solutions satisfy the desired properties, especially that of having small but non-zero W_0, becomes infeasible. One can simplify the problem somewhat by imposing discrete symmetries, as we discuss below, but this of course ignores most of the possible vacua.

[4] There are other examples in which the number of contributing divisors is infinite, such as the example of [66].

[5] This inequality can be violated for gauge-type divisors, but we believe all models containing these have $h^{1,1} > 1$. This point is discussed further in [45].

A simpler goal is to obtain information about the existence and number of flux vacua using the indirect approach of [53, 8, 69]. We want to know how many supersymmetric vacua we can expect with $L \leq L_*$, $e^{K_0}|W_0|^2 \leq \lambda_*$, the dilaton τ within a region $\hat{\mathcal{H}} \subseteq \mathcal{H}$ and the complex structure z within $\hat{\mathcal{M}}_c \subseteq \mathcal{M}_c$, in the limit of very small λ_*. By approximating the sum over flux lattice points by a volume integral in continuous flux space, this was computed in [8] to be

$$N_{\text{vac}} = \frac{(2\pi L_*)^{b_3}}{b_3!} \frac{\lambda_*}{L_*} \text{vol}(\hat{\mathcal{H}}) \int_{\hat{\mathcal{M}}_c} d^{2n}z \det g \, \rho_0(z) \tag{4.1}$$

where $b_3 = 2n+2$, g is the Weil-Petersson metric on $\mathcal{M}_c$, and ρ_0 a certain density function on $\mathcal{M}_c$ computed from local geometric data. The detailed expression for ρ_0 and a discussion of its evaluation for large n is given in appendix A. A useful estimate is the index density [53]:

$$\det g \, \rho_0 \sim \frac{1}{\pi^{n+1}} \det(R + \omega) \tag{4.2}$$

with R the curvature form and ω the Kähler form on $\mathcal{M}_c$. In particular $\rho_0 \neq 0$, so the distribution of vacua with $|W_0|^2 < \lambda_*$ is uniform in λ_* around $\lambda_* \sim 0$.

Performing the integral (4.1) over a region of moduli space provides an estimate for the number of quantized flux vacua in that region. While the estimate only becomes precise in the limit of large L_*, numerical experiments suggest it is fairly good for $L_* > b_3$. In this case, one expects a subregion of radius $r > \sqrt{b_3/L_*}$ with an expected number of vacua $N_{\text{vac}} \gg 1$ to contain flux vacua [8].

Once we know flux vacua exist in some region, there are better ways to find explicit flux vacua. We have developed a method which begins by fixing a rational point at large complex structure, finding the lattice of fluxes solving $DW = 0$ and finding short lattice vectors using advanced algorithms [70]. One can then move in by systematically correcting the point on moduli space to take into account the corrections from this limit. Since these corrections are small, this often produces vacua with small W_0, and we will cite some results obtained this way below.

4.1 Metastability

Although we will not study the question of whether one can break supersymmetry by antibranes or D terms in any detail, it leads to another constraint on the flux vacua which we will study: namely, we must insist that the potential is actually minimized at the candidate vacuum. This was not required for consistency of a supersymmetric AdS vacuum [71], and is trivial for a no-scale nonsupersymmetric vacuum, but this condition becomes non-trivial after size modulus stabilization.

As discussed in [8] and many other references, the mass matrix for a vacuum satisfying $DW = 0$, and for fields which do not participate in the D-type supersymmetry breaking, is

$$m^2 = H^2 - 3|\tilde{W}|H \,,$$

where we defined $\tilde{W} \equiv e^{K/2}|W|$, and the matrix

$$H = 2\, d^2|\tilde{W}| \,,$$

expressed in an orthonormal frame. Thus, positive eigenvalues of H which are less than $3|\tilde{W}|$ will lead to tachyons. In the KKLT construction, W_0 is assumed small, and we show below that this implies that $|W|$ is small, so that this need not be a stringent condition.

We again take the superpotential to be a sum

$$W = W_0(z,\tau) + W_\rho(\rho)\,. \tag{4.3}$$

The matrix H is given by

$$H = |\tilde{W}| \cdot \mathbf{1} + \frac{1}{|\tilde{W}|}\begin{pmatrix} 0 & S & 0 & T \\ \bar{S} & 0 & \bar{T} & 0 \\ 0 & T^t & 0 & U \\ \bar{T}^t & 0 & \bar{U} & 0 \end{pmatrix}$$

with

$$S = \bar{\tilde{W}} D_i D_j \tilde{W}\,, \qquad U = \bar{\tilde{W}} D_\alpha D_\beta \tilde{W}\,, \qquad T = \bar{\tilde{W}} D_i D_\beta \tilde{W}\,,$$

where i,j are orthonormal frame indices for the complex structure moduli and the dilaton, and α,β for the Kähler moduli.[6] If $S \gg U,T$, as is generically the case in the KKLT construction, the effect of T on the eigenvalues of H will be subleading. This expresses that when there is a large scale separation of complex and Kähler moduli masses, they can be treated separately. Moreover in general, if tachyons are found when T is set to zero (i.e. purely in complex structure or Kähler directions), there will also be tachyons in the full problem. Thus it is a good and useful approximation to put $T \equiv 0$, and we will do so in the following.

The condition on the complex structure moduli is then the same as in the discussion of [8] in which Kähler moduli were ignored, except that W is shifted to include W_ρ. This sets the overall scale of the condition on $D^2 W_{flux}$ but does not enter in the details. For the simplest model superpotential

$$W_\rho = b\, e^{i\rho/N}\,,$$

we have

$$W = \left(1 - \frac{1}{1 + 2\,\mathrm{Im}\,\rho/3N}\right) W_0$$

at $D_\rho W = 0$, so for typical ρ and N this decreases W and makes tachyons less likely, but not dramatically so.

For generic mass matrices, in a model with n complex structure moduli, some of these would be tachyonic in roughly a fraction $3n|W|$ of cases, which is small for small $|W|$.

[6] It may be counterintuitive that the mixing T is present as naively both K and W are the sum of two independent functions, so the two sectors appear to decouple. One way to see why this is naive is to note that this statement is not invariant under Kähler-Weyl transformations, even those which take the factorized form $W \to f_0(z,\tau) f_\rho(\rho) W$. A more mathematical way to say this is that the decomposition $K = K(z,\tau) + K(\rho)$ implies that W is a section of a tensor product line bundle, and in writing (4.3) one is implicitly choosing reference sections of the two bundles. The mixing matrix T is then $-|W|^2$ times the tensor product of the covariant derivatives of these reference sections.

Going on to consider the Kähler moduli, for one Kähler modulus, there are no tachyons if

$$g^{\rho\bar{\rho}}|D_\rho^2 W| > 2|W|$$

which is

$$\frac{4(\operatorname{Im}\rho)^2}{3}\left|\partial_\rho^2 W_\rho - \frac{3}{2i\operatorname{Im}\rho}\partial_\rho W_\rho - \frac{3}{4(\operatorname{Im}\rho)^2}W\right| > 2|W|\,.$$

For the model superpotential this becomes

$$\left|1+\frac{\operatorname{Im}\rho}{N}\right| > 1$$

which is always satisfied. This might fail in a multi-modulus model, though the examples we study below were found numerically to be tachyon-free as well.

Thus, in general one does not expect moduli to become tachyonic after supersymmetry breaking. However special structure in the mass matrix might change this conclusion, and the main point of this discussion is that (in the approximation that we ignore the one-loop determinants in W_ρ) the mass matrix for complex structure moduli can be analyzed in the simpler model in which Kähler moduli are simply left out of consideration, since their effect is just to renormalize W_0. This justifies the analysis of [53, 8] in which this was done.

In [8], the one parameter models were studied in great detail, and it was found that tachyons are generic in some regimes, for example near conifold points. This is potentially important as we might want to work near a conifold point to obtain a small scale or small supersymmetry breaking. The situation for multi-modulus models appears to depend on details of the specific model, and this might or might not be a problem.

5. Search for models

The upshot of the previous section is that a model which stabilizes Kähler moduli must be based on a fourfold X such that the divisors on Z whose pullbacks have arithmetic genus one form a "complete set" as discussed in subsection 3.3. The simplest way this can happen is if such divisors form a basis of $H_4(Z)$. We now show this is a rather strong requirement and that not very many models satisfy it.

There are two large classes of examples we consider: Fano threefolds and $\mathbb{P}^1$ bundles over toric surfaces.

First, since there is a classification theory of Fano threefolds, all such examples can be listed. In particular, [16] lists all the Fano threefolds together with the divisors that give rise to divisors of arithmetic genus one in the associated elliptically fibered fourfold.

Out of all the toric threefolds in the tables in [16], the models with a basis of divisors of arithmetic genus 1 are $B = \mathcal{F}_{11}$, $\mathcal{F}_{12}$, $\mathcal{F}_{14}$, $\mathcal{F}_{15}$, $\mathcal{F}_{16}$ and $\mathcal{F}_{18}$. The model with $B = \mathcal{F}_{17}$ comes close, but while it has enough fourfold divisors with arithmetic genus one, the corresponding divisors in the base do not generate the Picard group of $\mathcal{F}_{17}$. Thus, out of the 18 toric Fano manifolds in the list, only six work. We note that there are also 74 Fano threefolds that are not toric. Out of these, only 23 have enough fourfold divisors of arithmetic genus one.

We consider $\mathcal{F}_{18}$ and $\mathcal{F}_{11}$ in more detail below, because their orientifold limits have (relatively) few complex structure parameters.

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Another large class of examples can be obtained as $\mathbb{P}^1$ bundles over toric surfaces:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] & B \arrow[d, "\pi'"] \\ & B'. \end{tikzcd}$$

Let us start with the case $B' = \mathbb{P}^2$. The $\mathbb{P}^1$-bundle over B' will be specified by an integer n according to the following toric data:

$$\begin{array}{cccccc} & D_1 & D_2 & D_3 & D_4 & D_5 \\ \mathbb{C}^* & 1 & 1 & 1 & n & 0 \\ \mathbb{C}^* & 0 & 0 & 0 & 1 & 1 \end{array}\,. \tag{5.1}$$

$D_1, \ldots, D_5$ are the toric divisors: D_1, D_2, D_3 are the pullbacks of the three lines in $\mathbb{P}^2$ and D_4 and D_5 are the two sections of the $\mathbb{P}^1$ fibration. The arithmetic genus of the fourfold divisors $\mathcal{V}_i = \pi^*(D_i)$ is given by $\chi(\mathcal{V}_i, \mathcal{O}_{\mathcal{V}_i}) = 1/2 K_B D_i^2$, $i = 1, \ldots, 5$, where K_B is the canonical divisor of B [16]. An immediate computation gives

$$\chi(\mathcal{V}_1) = \chi(\mathcal{V}_2) = \chi(\mathcal{V}_3) = -1\,, \qquad \chi(\mathcal{V}_4) = -\frac{n(n+3)}{2}\,, \qquad \chi(\mathcal{V}_5) = \frac{n(-n+3)}{2}\,. \tag{5.2}$$

Now we see that that the pullbacks to the fourfold of the two sections can not have simultaneously arithmetic genus 1. However, let us first choose n to be 1 or -1 so that either $\mathcal{V}_4$ or $\mathcal{V}_5$ will have arithmetic genus 1. Now we can enforce an ADE type singularity along the section whose pullback does not contribute initially to the superpotential. The components of the singular fiber will have arithmetic genus 1 and will project to that section. Since the two sections are linearly independent, they generate the Picard group of B and we have thus obtained a model where the condition for stabilizing all the Kähler moduli is satisfied.

Note that in this case it is possible to stabilize the Kähler moduli using only divisors of gauge type, by enforcing ADE singularities of the Weierstrass model along both of the two sections of the $\mathbb{P}^1$ bundle. However, this will not be true for models with $h^{1,1} > 2$.

We can perform a similar analysis in the case when B' is a Hirzebruch surface $\mathbb{F}_m$; The toric threefold B is specified by two positive integers n and p as $B = \mathbb{P}(\mathcal{O}_{\mathbb{F}_m} \oplus \mathcal{O}_{\mathbb{F}_m}(nC_0 + pf))$, where C_0 and f are respectively the negative section and fiber of $\mathbb{F}_m$. The toric data of B is

$$\begin{array}{ccccccc} & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\ \mathbb{C}^* & 1 & 1 & m & 0 & p & 0 \\ \mathbb{C}^* & 0 & 0 & 1 & 1 & n & 0 \\ \mathbb{C}^* & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\,. \tag{5.3}$$

D_i, $i = 1, \ldots, 6$ denote again the toric divisors. Another quick computation gives

$$\begin{aligned} \chi(\mathcal{V}_1) &= 0\,, & \chi(\mathcal{V}_2) &= 0\,, \\ \chi(\mathcal{V}_3) &= -m\,, & \chi(\mathcal{V}_4) &= m\,, \\ \chi(\mathcal{V}_5) &= -n - p - np + \tfrac{mn(n+1)}{2}\,, & \chi(\mathcal{V}_6) &= n + p - np - \tfrac{mn(-n+1)}{2}\,. \end{aligned} \tag{5.4}$$

In order to satisfy the requirement stated above we need to have $m = 1$. It is easy to see that we can not have $\chi(\mathcal{V}_5) = \chi(\mathcal{V}_6) = 1$. These equations would imply that n and p

should satisfy $(n-p)^2 - p^2 = 2$ which however does not admit integer solutions. But since $\chi(\mathcal{V}_5)$ and $\chi(\mathcal{V}_6)$ are the the pullbacks of the two sections we can enforce again an ADE singularity of the Weierstrass model along one of them while choosing either $n = 2(p+1)$ or $n = 2(p-1)$ such that the pullback of the other section has also arithmetic genus 1. Obviously, in this case we can not construct a basis of the Picard group of B consisting only of arithmetic genus one divisors of gauge type, because $h^{1,1} = 3$.

A similar analysis can be carried in the case when B is a $\mathbb{P}^1$ bundle over a toric del Pezzo surface. By making specific choices for the data of the $\mathbb{P}^1$ bundle, in the case of the del Pezzo surfaces dP_2, dP_3 and dP_4, we can construct examples that have a full basis of divisors of instanton type contributing to the superpotential. What happens if we also consider divisors of gauge type? For any $\mathbb{P}^1$ bundle over dP_2 or dP_3, enforcing for example an ADE singularity along one of the sections, we obtain a model satisfying the above criterion. In the case of dP_4 we have to enforce singularities of the Weierstrass model along both sections of the $\mathbb{P}^1$ bundle.

To summarize, there are several models with toric Fano base in which instanton-type divisors can stabilize Kähler moduli. In these models, which can be analyzed using existing techniques, the presence of a suitable nonperturbative superpotential is clear.

There are also several possibilities for models with $\mathbb{P}^1$-fibered base in which gauge-type divisors can stabilize Kähler moduli. These models have heterotic duals and are potentially simpler, but establishing the existence of a suitable superpotential in these models requires controlling the matter content and matter superpotential of the gauge theories. This is a rather complicated problem in the F theory framework, which has not been solved in the detail we need; in particular the flux contributions to the matter superpotential are not yet well understood. Thus, we will not reach definite conclusions for these models in this work.

Finally, there are surely many more models in which the base is not Fano (any model with $h^{1,1} > 10$ is necessarily of this type), and there may also be models whose base has other fibration structures.

5.1 The $\mathcal{F}_{18}$ model

One of the examples from [16] is $B \equiv \mathcal{F}_{18}$, a toric Fano threefold [46, 20, 21]. By our previous discussion, Z will be a double cover of B, branched along its canonical divisor. Thus Z can be realized as a quadric in $Y = \mathbb{P}(\mathcal{O}_{\mathcal{F}_{18}} \oplus \mathcal{O}_{\mathcal{F}_{18}}(K_{\mathcal{F}_{18}}))$.

Note that Y is not a weighted projective space and its toric data is given by

$$\begin{array}{ccccccccccc}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & U & W \\
\mathbb{C}^* & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\
\mathbb{C}^* & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
\mathbb{C}^* & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
\mathbb{C}^* & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
\mathbb{C}^* & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\
\mathbb{C}^* & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
\mathbb{C}^* & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\
\mathbb{C}^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array} \,. \tag{5.5}$$

The generic divisor in the linear system $|-K_Y|$ is the smooth Calabi-Yau variety Z and is defined by the equation

$$W^2 + WU \sum_{(a_1,\ldots,a_8)'} f_{a_1\ldots a_8} \prod_{i=1}^{8} X_i^{a_i} + U^2 \sum_{(b_1,\ldots,b_8)'} g_{b_1\ldots b_8} \prod_{i=1}^{8} X_i^{b_i} = 0, \tag{5.6}$$

where the sums are taken over sets of positive integers $a_1,\ldots,a_8,b_1,\ldots,b_8$ which satisfy

$$\begin{aligned} -b_1+b_7+b_8 &= 2(-a_1+a_7+a_8) = 2\,, & b_1+b_2-b_4 &= 2(a_1+a_2-a_4) = 2\,,\\ -b_1+b_4+b_6 &= 2(-a_1+a_4+a_6) = 2\,, & b_1+b_3-b_6 &= 2(a_1+a_3-a_6) = 2\,,\\ -b_2+b_4+b_5 &= 2(-a_2+a_4+a_5) = 2\,, & b_2+b_3-b_5 &= 2(a_2+a_3-a_5) = 2\,,\\ -b_3+b_5+b_6 &= 2(-a_3+a_5+a_6) = 2\,. \end{aligned} \tag{5.7}$$

Using the invariance under reparametrizations, we can set $f_{a_1,\ldots,a_8} = 0$ and the hypersurface equation becomes

$$W^2 + U^2 \sum_{(b_1,\ldots,b_8)'} g_{b_1\ldots b_8} \prod_{i=1}^{8} X_i^{b_i} = 0\,. \tag{5.8}$$

The holomorphic involution $\hat{\Omega}$ is simply $W \to -W$. All the third cohomology of Z is odd under the (pull-back) of $\hat{\Omega}$ and in particular $\hat{\Omega}^*\Omega = -\Omega$, where Ω is the holomorphic three-form on Z. The third cohomology of the quotient $B = Z/\hat{\Omega}$ is trivial since B is toric. Conversely, $H^2(Z,\mathbb{Z})$ and $H^4(Z,\mathbb{Z})$ are even under $\hat{\Omega}^*$. Therefore, all the complex structure and Kähler deformations remain in the spectrum [44].

Thus, we obtain a IIB orientifold compactification on B of the type we want.

5.1.1 Topological analysis

This can be done using standard toric techniques [72]. The toric data of $\mathcal{F}_{18}$ is [19, 46, 20, 21]

$$\begin{array}{c|cccccccc} & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & D_8 \\ \mathbb{C}^* & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \mathbb{C}^* & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \mathbb{C}^* & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ \mathbb{C}^* & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ \mathbb{C}^* & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{C}^* & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \mathbb{C}^* & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \end{array}\,. \tag{5.9}$$

We choose D_i, $i = 1,\ldots,5$ as basis for $H^2(B,\mathbb{Z})$. We have the linear equivalence relations $D_6 = D_2 - D_3 + D_4, D_7 = D_8 = -D_1 + D_3 - D_4 + D_5$.

The Kähler cone is not simplicial: although it is five dimensional, it has six generators which are given by

$$\begin{aligned} \mathcal{R}_1 &= D_7\,, & \mathcal{R}_2 &= D_1 + D_6 + D_7\,,\\ \mathcal{R}_3 &= D_1 + D_3 + D_6 + D_7\,, & \mathcal{R}_4 &= D_3 + D_6\,,\\ \mathcal{R}_5 &= D_1 - D_2 + D_3 + D_6 + D_7\,, & \mathcal{R}_6 &= D_1 - D_2 + D_3 + 2D_6 + D_7\,. \end{aligned} \tag{5.10}$$

In order to define a large radius limit and Kähler coordinates, we need to choose a simplicial decomposition of the Kähler cone and pick one of the subcones [73]. We note that there exists a simplicial decomposition such that one of the subcones is generated by $\mathcal{R}_j$, $j = 1, \ldots, 5$ and therefore we take the Kähler form to be

$$J = \sum_{j=1}^{5} t_j \mathcal{R}_j = (t_2 + t_3 + t_5) D_1 - t_5 D_2 + (t_3 + t_4 + t_5) D_3 + (t_2 + t_3 + t_4 + t_5) D_6 + \\ + (t_1 + t_2 + t_3 + t_5) D_7 \,. \tag{5.11}$$

The dual polyhedron ∇_Z which encodes the divisors of the Calabi-Yau threefold Z has the following vertices:

$$\begin{array}{lllll} (0,0,-1,1)\,, & (0,1,1,1)\,, & (0,-1,0,1)\,, & (0,1,0,1)\,, & (0,0,1,1), \\ (0,-1,-1,1)\,, & (1,0,0,1)\,, & (-1,0,-1,1)\,, & (0,0,0,-1)\,. & \end{array} \tag{5.12}$$

The Hodge numbers[7] of Z are $h^{1,1}(Z) = 5$, $h^{2,1}(Z) = 89$.

Next, we need to know how divisors of B pull back to the Calabi-Yau fourfold X. Since X elliptically fibered over the base $B = \mathcal{F}_{18}$, we can construct it as a hypersurface in a toric variety.

Using this, we can construct the dual polyhedron ∇_X, which encodes the divisors of the Calabi-Yau fourfold X. It has vertices:

$$\begin{array}{lllll} (0,0,-1,2,3)\,, & (0,1,1,2,3)\,, & (0,-1,0,2,3)\,, & (0,1,0,2,3)\,, & (0,0,1,2,3)\,, \\ (0,-1,-1,2,3)\,, & (1,0,0,2,3)\,, & (-1,0,-1,2,3)\,, & (0,0,0,-1,0)\,, & (0,0,0,0,-1)\,. \end{array} \tag{5.13}$$

Standard toric methods give for the Hodge numbers of the fourfold $h^{1,1} = 6$, $h^{2,1} = 0$, $h^{3,1} = 2194$, $h^{2,2} = 8844$, $\chi = 13248$. We note that it is possible to find a triangulation of ∇_X consistent with its elliptic fibration structure such that each of the top dimensional cones having unit volume, thus guaranteeing smoothness of the corresponding Calabi-Yau fourfold.

5.1.2 Nonperturbative superpotential

The fibration $\pi : X \longrightarrow B$ has a section $\sigma : B \longrightarrow X$, $\pi \circ \sigma = \mathbb{1}_B$. The following toric divisors have arithmetic genus one: $\mathcal{V}_1 = (0,0,-1,2,3)$, $\mathcal{V}_2 = (0,1,1,2,3)$, $\mathcal{V}_3 = (0,-1,0,2,3)$, $\mathcal{V}_4 = (0,1,0,2,3)$, $\mathcal{V}_5 = (0,0,1,2,3)$, $\mathcal{V}_6 = (0,-1,-1,2,3)$ and $\Sigma = (0,0,0,2,3)$. The first six divisors are vertical, $\mathcal{V}_i = \pi^*(D_i)$, $i = 1, \ldots, 6$, and therefore contribute to the F-theory superpotential, while Σ does not contribute, since is the section of the elliptic fibration, $\Sigma = \sigma(B)$. Moreover, since $\mathcal{V}_j$, $j = 1, \ldots, 6$ are vertices of ∇_X, $h^{0,i}(\mathcal{V}_j) = 0$, $i = 1,2,3, j = 1, \ldots, 6$. Now, since $D_1, \ldots, D_5$ generate $H^2(B, \mathbb{Z})$, we see that this model has a complete set of contributing divisors and satisfies the requirement of section 3.

Note that the divisor D_1 on the base B is the exceptional divisor that corresponds to the contractions of both first and sixth Mori cone generators. This is possible since D_1 is isomorphic to a Hirzebruch surface $\mathbb{F}_0$ which is a product of those two curves.

In order to study the question of fixing the Kähler moduli, we need to compute the volumes of the divisors contributing to the superpotential, as well the volume of the three

[7] We have used the computer program POLYHEDRON, written by Philip Candelas.

dimensional base B. To achieve that, we triangulate the fan of B. We list the cones below:

$$\begin{aligned}
&D_1D_4D_7\,, \quad D_1D_4D_8\,, \quad D_1D_6D_7\,, \quad D_1D_6D_8\,, \quad D_2D_4D_7\,, \quad D_2D_4D_8\,,\\
&D_2D_5D_7\,, \quad D_2D_5D_8\,, \quad D_3D_5D_7\,, \quad D_3D_5D_8\,, \quad D_3D_6D_7\,, \quad D_3D_6D_8\,.
\end{aligned} \tag{5.14}$$

Using the above (unique) triangulation, we obtain the following nonvanishing intersection numbers

$$\begin{aligned}
D_1^3 &= 2\,, & D_1^2D_4 &= -1\,, & D_1^2D_6 &= -1\,, & D_1^2D_7 &= -1\,, & D_2^3 &= -1\,,\\
D_2^2D_4 &= 1\,, & D_2^2D_7 &= -1\,, & D_3^3 &= -1\,, & D_3^2D_6 &= 1\,, & D_3^2D_7 &= -1\,,\\
D_4^2D_2 &= -1\,, & D_4^3 &= 1\,, & D_4^2D_7 &= -1\,, & D_5^2D_2 &= 1\,, & D_5^2D_3 &= 1\,,\\
D_5^3 &= -2\,, & D_5^2D_7 &= -1\,, & D_6^2D_3 &= -1\,, & D_6^3 &= 1\,, & D_6^2D_7 &= -1\,.
\end{aligned} \tag{5.15}$$

We obtain for the divisor volumes $\tau_i \equiv D_iJ^2/2$ and the total volume $V \equiv J^3/6$:

$$\begin{aligned}
\tau_1 &= t_1t_4\,, & \tau_2 &= \frac{t_5}{2}(2t_1+2t_2+2t_3+t_5)\,,\\
\tau_3 &= \frac{t_2}{2}(2t_1+t_2+2t_3+2t_5)\,, & \tau_4 &= \frac{1}{2}(t_2+t_3)(2t_1+t_2+t_3)\\
\tau_5 &= (t_3+t_4)(t_1+t_2+t_3+t_5)\,, & \tau_6 &= \frac{1}{2}(t_3+t_5)(2t_1+t_3+t_5)\,,\\
V &= t_1t_2t_3+\frac{t_2^2t_3}{2}+\frac{t_1t_3^2}{2}+t_2t_3^2+\frac{t_3^3}{3}+t_1t_2t_4+\frac{t_2^2t_4}{2}+t_1t_3t_4+t_2t_3t_4+\frac{t_3^2t_4}{2}+\\
&\quad +t_1t_2t_5+\frac{t_2^2t_5}{2}+t_1t_3t_5+2t_2t_3t_5+t_3^2t_5+t_1t_4t_5+t_2t_4t_5+t_3t_4t_5+\frac{t_2t_5^2}{2}+\\
&\quad +\frac{t_3t_5^2}{2}+\frac{t_4t_5^2}{2}\,.
\end{aligned} \tag{5.16}$$

5.1.3 Complex structure moduli

To compute or count flux vacua in arbitrary regions of the complex structure moduli space of the Calabi-Yau threefold Z, one would have to compute the periods for the generic hypersurface Z. These are generalized hypergeometric functions in 89 variables. In principle they can be computed using existing techniques, but this would require a lot of work, even using a computer.

A somewhat easier (but still formidable) task is to describe the periods in the vicinity of the large complex structure limit. This is equivalent to computing the triple intersections for the mirror threefold $\tilde{Z}$, which has $h^{2,1}=5$ and $h^{1,1}=89$. To make the description of this part of the moduli space complete, one furthermore has to compute the Kähler cone, i.e. the part of the parameter space $\mathbb{R}^{89}$ where all holomorphic curves have positive area. We describe the algorithms we used to achieve this in appendix B.

5.1.4 Flux vacua

Counting flux vacua can be done using the techniques of [53, 8] as outlined in section 4. Since $L_* = 13248/24 = 552 \gg b_3 = 180$, we expect that the approximations made to derive the counting formula (4.1) should be valid.

According to this formula, the expected number of vacua with cosmological constant $e^{K_0}|W_0|^2$ less than $\lambda_* \ll L_*$, equals $(2\pi L)^{b_3}/b_3! \sim 10^{307}$ multiplied by λ_*/L_* times the integral of a geometrical density function. Taking the integration domain $\hat{\mathcal{M}}_c$ equal to the entire moduli space, the estimate (4.2) indicates that the geometrical factor should be of the order of the Euler characteristic of $\mathcal{M}_c$. This does not need to be an integer, as the moduli space is a noncompact orbifold, and in examples [8] is a small fraction, of order $1/|\Gamma|$, where Γ is the order of a finite group (e.g. $\mathbb{Z}_5$ for the mirror quintic) or the volume of a group [74]. This will be far larger than 10^{-307} and thus we expect many vacua with very small cosmological constant.

Can we find such vacua at large complex structure, or equivalently, at large volume of the mirror? We can judge this by integrating the density function over, say, the region defined by requiring all curve areas of the mirror to be bigger than 1. Using the approach for estimating the density function ρ_0 explained in appendix A and the construction of the geometry of the moduli space outlined in appendix B, we have done Monte Carlo estimates of this integral. The results are as follows. The average value of μ defined in appendix A is of order 10^{90}, hence the density function ρ_0 is of order 10^{240}, and we take this out of the integral (4.1). The remaining volume integral, evaluated using 10^7 Monte Carlo sample points, gives a number of order 10^{-650}. Putting everything together we get, up to the factor λ_*/L_*,

$$N_{\mathrm{vac}}(LCS) \sim 10^{-100} \,. \tag{5.17}$$

Why is the volume so small? One important reason is the mirror volume suppression factor $\tilde{V}^{-n/3}$ mentioned at the end of appendix A. For curve areas y^i bigger than one, the mirror volume $\tilde{V} > 10^{10}$. This becomes understandable when one considers that when $\tilde{V}$ is written in terms of the curve area coordinates y^i (cf. appendix A), the expression $\tilde{V} = \tilde{D}^{(y)}_{ijk} y^i y^j y^k/6$ contains $\sim 10^6$ terms, and the $\tilde{D}^{(y)}_{ijk}$ are widely distributed between 0 and 10^8. With $\tilde{V} > 10^{10}$, the volume suppression factor is $\sim 10^{-300}$. Additional suppression comes from the fact that $\det g \sim 10^{-100}$ for typical values of y with $\tilde{V}(y) = 1$, and from the smallness of the euclidean volume of the surface $\tilde{V}(y) = 1$. It is possible that the Monte Carlo missed regions in which $\tilde{V}$ is smaller than 10^{10}, or that some of this is an artifact of our approximate parametrization of the Kähler cone with the curve areas y^i, but we see no particular evidence for this.

Thus it seems likely that any flux vacua in this region are special cases, and there is no reason to expect a multiplicity of vacua out of which some would have small W_0. This paucity of vacua in the large complex structure limit is not specific to this example, but rather is a very general feature of models with many moduli, as explained in [75].

If we neglect world-sheet instanton corrections, then by lowering the cutoff to $y^i > 0.075$, the expected number of vacua becomes of order 1. In few modulus examples, the instanton sums tend to converge all the way down to zero, so this might be a valid indication of where vacua will start to exist.

5.2 The $\mathcal{F}_{11}$ model

Another example from Grassi's list is the base $B = \mathcal{F}_{11}$ [20, 21]. We now construct an explicit toric model of a Calabi-Yau fourfold elliptically fibered over this base. This model has an orientifold limit Z with $h^{1,1} = 3, h^{2,1} = 111$.

The toric data of $\mathcal{F}_{11}$ is given by [19, 46, 20, 21]

$$\begin{array}{ccccccc} & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\ \mathbb{C}^* & 0 & -2 & 1 & 1 & 1 & 0 \\ \mathbb{C}^* & -1 & 0 & 1 & 0 & 0 & 1 \\ \mathbb{C}^* & 1 & 1 & -1 & 0 & 0 & 0 \end{array} . \tag{5.18}$$

The generators of the Kähler cone of B are

$$\mathcal{R}_1 = D_5\,, \qquad \mathcal{R}_2 = D_6\,, \qquad \mathcal{R}_3 = D_1 + D_6\,. \tag{5.19}$$

We have the linear equivalence relations $D_4 = D_5 = D_1 + D_3$ and $D_6 = D_1 + D_2 + 2D_3$. The Calabi-Yau fourfold elliptically fibered over B may be constructed as a hypersurface in a toric variety. The dual polyhedron ∇, which encode the divisors of the Calabi-Yau fourfold has vertices:

$$\begin{array}{lllll} (1,1,0,2,3)\,, & (0,-1,0,2,3)\,, & (1,0,0,2,3)\,, & (0,-1,1,2,3)\,, & (-1,-1,-1,2,3), \\ (0,1,0,2,3)\,, & (0,0,0,-1,0)\,, & (0,0,0,0,-1)\,. & & \end{array}$$

The Hodge numbers of this fourfold are $h^{1,1} = 4$, $h^{2,1} = 0$, $h^{3,1} = 3036$, $h^{2,2} = 12204$, $\chi = 18288$. Again, we find that the fourfold is smooth.

The fibration $\pi : X \longrightarrow B$ has a section $\sigma : B \longrightarrow X$, $\pi \circ \sigma = \mathbb{1}_B$. The following divisors have arithmetic genus one: $\mathcal{V}_1 = (1,1,0,2,3)$, $\mathcal{V}_2 = (0,-1,0,2,3)$, $\mathcal{V}_3 = (1,0,0,2,3)$ and $\Sigma = (0,0,0,2,3)$. The first three divisors are vertical, $\mathcal{V}_i = \pi^*(D_i)$, $i = 1,2,3$, and therefore may contribute to the F-theory superpotential, while Σ does not contribute, since is the section of the elliptic fibration, $\Sigma = \sigma(B)$. It is possible to check that in fact $h^{0,i}(\mathcal{V}_j) = 0$, $i,j = 1,2,3$, thus all the vertical divisors do give a contribution to the superpotential. Therefore, this model also provides a complete basis of divisors.

To compute the volumes of the divisors D_1, D_2, D_3, we triangulate the fan of B. Its 3-dimensional cones are $D_1D_4D_6$, $D_1D_5D_6$, $D_4D_5D_6$, $D_2D_4D_5$, $D_1D_3D_4$, $D_1D_3D_5$, $D_2D_3D_4$, $D_2D_3D_5$. Using this (unique) triangulation, we obtain the following nonvanishing triple intersections:

$$D_1^3 = -3\,, \quad D_1^2 D_3 = 2\,, \quad D_1 D_3^2 = -1\,, \quad D_2^3 = 4\,, \quad D_2^2 D_3 = -2\,, \quad D_2 D_3^2 = 1\,. \tag{5.20}$$

Let $J = \sum_{i=1}^3 t_i \mathcal{R}_i$ be the Kähler form of B. We obtain for the divisor volumes $\tau_i \equiv D_i J^2/2$ and the total volume $V \equiv J^3/6$:

$$\begin{aligned} \tau_1 &= \frac{t_2}{2}(2t_1 + t_2 + 4t_3)\,, \\ \tau_2 &= \frac{t_1^2}{2}\,, \qquad \tau_3 = t_3(t_1 + t_3)\,, \\ V &= \frac{t_1^2 t_2}{2} + \frac{t_1 t_2^2}{2} + \frac{t_2^3}{6} + \frac{t_1^2 t_3}{2} + 2t_1t_2t_3 + t_2^2 t_3 + t_1 t_3^2 + 2t_2 t_3^2 + \frac{2t_3^3}{3}\,. \end{aligned} \tag{5.21}$$

5.3 The orientifold of $\mathbb{P}^4_{[1,1,1,6,9]}$

Finally, out of the various possibilities which can be obtained as $\mathbb{P}^1$ bundles over a toric surface, the simplest is perhaps the $\mathbb{P}^1$ bundle over $\mathbb{P}^2$. The toric data of the threefold is presented in (5.1), where we take $n = -6$, so that $B = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-6))$. The toric data for B is as follows:

$$\begin{array}{cccccc} & D_1 & D_2 & D_3 & D_4 & D_5 \\ \mathbb{C}^* & 1 & 1 & 1 & -6 & 0 \\ \mathbb{C}^* & 0 & 0 & 0 & 1 & 1 \end{array}. \tag{5.22}$$

The toric divisors D_4 and D_5 are the sections of the $\mathbb{P}^1$ bundle over $B' = \mathbb{P}^2$. We will construct an elliptic fourfold X over B in such a way that X will be the resolution of a Weierstrass model W which has a D_4 singularity along the first section and an E_6 singularity along the second section. The motivation for this choice will become clear later, when we study Kähler moduli stabilization for these models

The dual polyhedron ∇_X, which encode the divisors of the Calabi -Yau fourfold has vertices:

$$\begin{array}{lllll} (-1,-1,6,2,3)\,, & (0,0,-3,2,3)\,, & (0,1,0,2,3)\,, & (1,0,0,2,3)\,, & (0,0,-2,1,1)\,, \\ (0,0,-1,0,0)\,, & (0,0,0,-1,0)\,, & (0,0,0,0,-1)\,. & & \end{array}$$

The Hodge numbers of this fourfold are $h^{1,1} = 13$, $h^{2,1} = 0$, $h^{3,1} = 1071$, $h^{2,2} = 4380$, $\chi = 6552$. Again, we find that the fourfold is smooth. We can also enforce an E_7 or E_8 singularity along the infinity section. The data for the fourfolds obtained this way are, respectively, $h^{1,1} = 14$, $h^{2,1} = 0$, $h^{3,1} = 935$, $h^{2,2} = 3840$, $\chi = 5742$ and $h^{1,1} = 16$, $h^{2,1} = 253$, $h^{3,1} = 745$, $h^{2,2} = 2582$, $\chi = 3096$.

It is easy to check that in this case the orientifold limit Z will be an elliptic fibration over $\mathbb{P}^2$, which is familiar as the hypersurface in weighted projective space $\mathbb{P}^4_{[1,1,1,6,9]}$ studied in [47, 76] and several other works. In order to do that, we note that Z is given by a quadric in a toric variety Y described by the following data

$$\begin{array}{cccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & U & W \\ \mathbb{C}^* & 1 & 1 & 1 & -6 & 0 & 3 & 0 \\ \mathbb{C}^* & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ \mathbb{C}^* & 0 & 0 & 0 & 0 & 0 & 1 & 1. \end{array} \tag{5.23}$$

This threefold has $h^{1,1} = 2$ and $h^{2,1} = 272$. There is a unique toric Calabi-Yau threefold with this Hodge numbers, the elliptic fibration over $\mathbb{P}^2$ [77].

Nonperturbative superpotential. We proceed as before and start by listing all toric divisors of arithmetic genus one for the first of the models above. These are: $\mathcal{V}_1 = (0,0,-3,2,3)$, $\mathcal{V}_2 = (0,0,-2,1,1)$, $\mathcal{V}_3 = (0,0,-2,1,2)$, $\mathcal{V}_4 = (0,0,-2,2,3)$, $\mathcal{V}_5 = (0,0,-1,0,0)$, $\mathcal{V}_6 = (0,0,-1,0,1)$, $\mathcal{V}_7 = (0,0,-1,2,3)$, $\mathcal{V}_8 = (0,0,2,2,3)$, $\mathcal{V}_9 = (0,0,1,2,3)$.

The first seven divisors project to D_5 and the last two project to D_4. These are going to be the only base divisors contributing to the superpotential for the other two models as well.

To compute the volumes of the divisors D_4 and D_5 we start by triangulating the fan of B. Its three dimensional cones are $D_1D_2D_4$, $D_1D_2D_5$, $D_1D_3D_4$, $D_1D_3D_5$, $D_2D_3D_4$ and $D_2D_3D_5$. Using this triangulation we obtain the following nonzero intersection numbers

$$\begin{array}{llllll} D_1^2D_4 = 1, & D_1^2D_5 = 1, & D_2^2D_4 = 1, & D_2^2D_5 = 1, & D_3^2D_4 = 1, & D_3^2D_5 = 1, \\ D_1D_4^2 = -6, & D_1D_5^2 = 6, & D_2D_4^2 = -6, & D_2D_5^2 = 6, & D_3D_4^2 = -6, & D_3D_5^2 = 6, \\ D_4^3 = 36, & D_5^3 = 36. & & & & \end{array} \tag{5.24}$$

Let $J = t_1D_1 + t_5D_5$ be the Kähler form of B. We obtain for the divisor volumes $\tau_i \equiv D_iJ^2/2$ and the total volume $V \equiv J^3/6$:

$$\begin{aligned} \tau_4 &= \frac{t_1^2}{2}, \qquad \tau_5 = \frac{(t_1+6t_5)^2}{2}, \\ V &= \frac{1}{6}(3t_1^2t_5 + 18t_1t_5^2 + 36t_5^3)\,. \end{aligned} \tag{5.25}$$

Complex structure moduli and flux vacua. The full 272 parameter prepotential for this model has not been worked out. However, this moduli space admits a $\Gamma \equiv \mathbb{Z}_6 \times \mathbb{Z}_{18}$ action, which fixes the two parameter subspace of CY's with defining equation

$$f = x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 - 18\psi x_1x_2x_3x_4x_5 - 3\phi x_1^6x_2^6x_3^6\,.$$

This is the subset of CY's obtained by the mirror construction of [78] and the six periods of these mirror CY's, which is the same as the subset of periods of Γ-invariant cycles, are worked out in [47].

If one turns on flux only on these cycles, since the resulting superpotential and Kähler potential are Γ-invariant, one is guaranteed that $D_iW = 0$ in all Γ-noninvariant directions, and thus one can find flux vacua just by working in the Γ-invariant part of the moduli space, call this $\mathcal{M}_\Gamma$. (This observation is also made in [67].) Of course, one would eventually need to check that all moduli remain non-tachyonic after supersymmetry breaking.

For the choices of gauge groups we considered, $L = \chi/24 \sim 100 - 300$, so the total index for flux vacua in this subspace is [53, 8]

$$I = \frac{(2\pi L)^6}{6!}\int_{\mathcal{M}_\Gamma\times\mathcal{H}} \det(-R-\omega)\,.$$

As mentioned earlier, the integral can be estimated as $1/12|\Gamma| = 1/1296$, leading to

$$I \sim 10^{11} - 10^{13}\,.$$

The number at weak coupling $1/g_s^2 < \epsilon$ will be roughly ϵ times this, and the number with $e^{K_0}|W_0|^2 < \lambda LT_3$ will be roughly λ times this, so there are clearly many weakly coupled flux vacua with small W_0.

Just to get a few explicit flux vacua, we consider the region of large complex structure, i.e. the region in which instanton corrections are small, $N_ie^{2\pi iw_i} \ll 1$. From [47], the prepotential at third order in the world-sheet instanton expansion is

$$\begin{aligned}\mathcal{F} = & \frac{1}{6}\left(9w_1^3 + 9w_1^2 w_2 + 3w_1 w_2^2\right) - \frac{9}{4}w_1^2 - \frac{3}{2}w_1 w_2 - \frac{17}{4}w_1 - \frac{3}{2}w_2 + \xi + \\ & + \frac{1}{(2\pi i)^3}\Bigg(540 q_1 + 3 q_2 + \frac{1215}{2}q_1^2 - 1080 q_1 q_2 - \frac{45}{2}q_2^2 + \\ & \qquad + 560 q_1^3 + 143370 q_1^2 q_2 + 2700 q_1 q_2^2 + \frac{244}{9}q_2^3 + \cdots\Bigg)\end{aligned} \tag{5.26}$$

with $q_i \equiv e^{2\pi i w_i}$ and $\xi \equiv \frac{\zeta(3)\chi(Z)}{2(2\pi i)^3} \approx -1.30843\,i$.

Looking at the coefficients in this expansion, the instanton corrections should be small for $w_2 \gg 1/6$ and $w_1 > 1$. To find quantized flux vacua, we used the procedure discussed in section 4. As an example, we first look for fluxes which stabilize the moduli at the rational point $\tau = 3i, w_1 = i, w_2 = i$, ignoring world-sheet instanton corrections and rationally approximating ξ by $-13/10$. One finds a quantized flux vacuum with $W = 0$, with fluxes $(N_{RR}; N_{NS})$ equal to

$$\{0, 69, 28, 0, 0, -20; -49, -18, -6, -4, 0, 0\}\,; \qquad L = 352$$

granting that the quantization condition on the orientifold is the same as on the original CY. This was argued in general in [33]; the fact that some cycles are smaller on the orientifold which naively doubles the Dirac quantum, is compensated for by the possibility of discrete RR and NS flux at the orientifold fixed points. A careful discussion in the present example might be possible using K theory [79].

Restoring the exact value of ξ, including the first instanton corrections and solving the resulting equations $DW = 0$ produces a $W \neq 0$ vacuum near the starting point. We find

$$\begin{aligned}\tau &= 2.945i \\ w_1 &= 0.9625i \\ w_2 &= 1.1037i \\ e^K|W|^2 &= 1.379 \times 10^{-4}\,.\end{aligned}$$

Examples with larger τ, w_i can easily be found in this way, but L tends to become bigger then as well.

For these vacua, one can check that the instanton corrections are small (as one would guess by the small corrections they lead to for the moduli) and the vacua appear sound, on the level we are working.

6. Numerical results on Kähler stabilization

6.1 $\mathcal{F}_{18}$ model

The Kähler moduli are stabilized for generic order 1 values of the b_i. Taking $W_0 = 10^{-30}$, the typical values for the t_i are $t_1 \sim 50 - 100$, $t_i \sim 0.1 - 0.3$ for $i = 2, 4, 5$ and $t_3 \sim 0$. The corresponding values of the contributing divisor volumes are $V_i \sim 11 - 12$, and the total volume $V \sim 5 - 10$. Taking the b_i all equal gives a nongeneric singular solution: $t_1 = \infty$, $t_i = 0$ for $i > 0$. But with $b_i = (1, 1.5, 2, 2.5, 3, 3.5)$ for example, we get $t_i = (53.3, 0.209, 0.00156, 0.208, 0.222)$, $\tau_i = (11.1, 11.9, 11.2 + i\pi, 11.2, 11.3 + i\pi, 11.9 + i\pi)$, and $V = 7.32$. Different choices for the phases of W_0 and five of the b_i can be absorbed in shifts of the imaginary parts of the τ_i (the axions). Note that since the areas of the Mori cone generators are $(t_1, t_2 + t_3, t_4 + t_3, t_5 + t_3, t_5, t_4, t_2)$, these solutions lie well inside the Kähler cone; $t_3 = 0$ is only an interior boundary of a subsimplex of the Kähler cone.

The typical values for the t_i and the total volume are rather small for this model, so α' corrections could become important. (The situation is better for the other models we consider.) However, since the solution in terms of the τ_i is mainly determined by the exponential factors (hence the near-equal values of $\tau_i \sim -(\ln W_0)/2\pi \approx 11$), it is reasonable to believe that it will still exist at approximately the same values of τ_i even after taking into account such corrections.

Finally, to have a stable vacuum after lifting the cosmological constant to a positive value, the critical point has to be a minimum of the potential. We verified numerically that this is indeed the case.

6.2 $\mathcal{F}_{11}$ model

The Kähler moduli are stabilized for generic order 1 values of the b_i. For $W_0 = -10^{-30}$, $b_i = 1$, we get $t_i = (4.89, 1.30, 1.76)$. These are also the areas of the three generators of the Mori cone. The corresponding complexified volumes of the divisors D_i contributing to the superpotential are $\tau_i = (11.8, 11.9, 11.7)$ and the total volume is $V = 93.3$. The critical point is a minimum of the potential.

6.3 $\mathbb{P}^4_{[1,1,1,6,9]}$ model

Assuming the gauge theory generates a superpotential $W = W_0 + \sum_{i=1}^{2} b_i e^{-2\pi a_i \tau_i}$, where τ_1, τ_2 are the complexified volumes of the divisors D_4 and D_5, we find that the Kähler moduli are stabilized for generic order 1 values of the b_i, provided $a_1 > a_2$. The following are some of the values of t_i, V_i and the volume V, obtained for $b_i = 1$ and different choices of a_1, a_2 and W_0:

a_1	a_2	W_0	t_1	t_5	V_1	V_2	V
1/4	1/30	10^{-30}	9.83	2.76	48.3	348	484
1/4	1/30	10^{-5}	4.61	1.14	10.6	65.7	39.1
1/4	1/12	10^{-30}	9.73	1.16	47.4	139	103
1/4	1/12	10^{-5}	4.40	0.468	9.64	25.9	8.01

(6.1)

The chosen values of a_i correspond to pure $G_2 \times E_8$ resp. $G_2 \times E_6$ gauge theory. Again, the critical point is a minimum of the potential.

The fact that $a_1 > a_2$ is needed to have a solution (lying well inside the Kähler cone) can be seen directly from (5.25): the approximate solution is $\tau_i = -\ln W_0 / 2\pi a_i$, but (5.25) implies $\tau_1 < \tau_2$, so $a_1 > a_2$.

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7. Conclusions

Our primary result is that we have candidate IIb orientifold compactifications in which nonperturbative effects will stabilize all complex structure, Kähler and dilaton moduli.

Models which stabilize all Kähler moduli by D3-instanton effects are not generic, but not uncommon either. We listed all the possibilities with Fano threefold base, and many possibilities whose bases are $\mathbb{P}^1$ fibrations. It turns out that these models must have several Kähler moduli and several non-perturbative contributions to the superpotential, as in the early racetrack scenarios for moduli stabilization.

We see no obstacle to adding supersymmetry breaking effects such as the antibrane suggested by KKLT, D term effects or others. One also expects these potentials to have many F breaking minima (statistical arguments for this are given in [8]). On general grounds, since the configuration spaces parameterized by other moduli such as brane and bundle moduli are compact, after supersymmetry breaking all moduli should be stabilized.

The models with Fano threefold base are rather complicated, with many complex structure moduli. It is not clear to us that this makes them less likely candidates to describe real world physics, but it is certainly a problem when using them as illustrative examples.

There may well be simpler models among the $\mathbb{P}^1$ fibered models, whose nonperturbative effects have a simple gauge theory picture. Perhaps the simplest is the $\mathbb{P}^1$ fibration over $\mathbb{P}^2$ or "11169 model." While the standard F theory analysis of the models we discussed have suggests that they have too much massless matter in one gauge factor to produce non-perturbative superpotentials, we suspect that other effects, in particular flux couplings to these brane world-volumes, would lift this matter, leading to working models, and intend to return to this in future work.

Another possibility for finding simpler models would be to stabilize some of the Kähler moduli with D terms. Following lines discussed in [80], one can show that in configurations containing branes wrapping k distinct cycles, D terms will generally stabilize $k-1$ relative size moduli. Since this relies on α' corrections, the resulting values of Kähler moduli will be string scale, but order one factors in the volumes and gauge field strengths might be arranged to make this a few times the string scale, which could suffice.

In any case, even the simplest models under discussion have many more moduli to stabilize. Furthermore, one must check that a candidate vacuum is not just a solution, but has no tachyonic instability. Now once one has established that these moduli indeed parameterize a compact configuration space, it is clear that the minimum of the effective potential on this space will be a stable vacuum. While it may be hard to compute the value of the potential at this minimum, the large number of flux vacua strongly suggests (as in [30]) that whatever it is, it can be offset by a flux contribution to lead to a metastable de Sitter vacuum. Thus, granting the effective potential framework, it will be extremely surprising if vacua of this type do not exist, while technically quite difficult to find them. This is the type of picture which motivates the statistical approach discussed in [81, 53, 23, 8, 75], as well as anthropic considerations such as [82, 83, 23].

Can we say anything about the validity of effective theory? At present we see no clear reason from string/M-theory or quantum gravity to doubt it. However, even within the effective field theory framework, there is another important assumption in KKLT, our work, and the other works along these lines. Namely, we have done Kaluza-Klein reduction in deriving the configuration space of Calabi-Yau metrics, and typically will take similar steps throughout the derivation of the effective field theory. Could it be that in many of these backgrounds, some of the KK and stringy modes which are dropped in this analysis are in fact tachyonic? Since these constructions rely on approximate cancellations between many diverse contributions to the vacuum energy, it is conceivable that subsectors of the theory have instabilities which do not show up in the final effective lagrangian; this should be examined. In any case, a lot of work remains to see whether these models are as plentiful as they now appear.

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A. Counting flux vacua when b_3 is large

Equation (4.1) gives the expected number of flux vacua with $L \leq L_*$, $|W_0|^2 \leq \lambda_*$, $\tau \in \hat{\mathcal{H}} \subseteq \mathcal{H}$ and $z \in \hat{\mathcal{M}}_c \subseteq \mathcal{M}_c$, for $\lambda_* \ll L_*$. Here we will derive expressions for ρ_0 suitable for Monte Carlo estimates in the case of many moduli.

It is somewhat more convenient to rewrite (4.1) first as

$$N_{\rm vac} = {\rm vol}(\hat{\mathcal{H}}) \int_{\hat{\mathcal{M}}_c} d^{2n}z \, \det g \, \nu(z) \,. \tag{A.1}$$

The density function $\nu(z)$ is [8]:

$$\nu(z) = \frac{2^{2n+2}}{\det g} \int_{|X|^2 \leq \lambda_*} d^2X \int_{Z_i Z^i \leq L_*} d^{2n}Z \left| \det \begin{pmatrix} 0 & Z_j \\ Z_i & \mathcal{F}_{ijk}\bar{Z}^k \end{pmatrix} \right|^2 + \mathcal{O}(X) \,. \tag{A.2}$$

The integration variables X and Z^i range over $\mathbb{C}$ and are the remnants of the flux vector N after a certain z-dependent change of basis and after imposing the constraint $DW_0 = 0$ (now considered as a constraint on the continuous flux components at a given point z). Indices are lowered with the metric $g_{i\bar{j}}$. The factor in front of the integral comes from the jacobian corresponding to the change of basis, and the determinant in the integrand is the $|\det D^2 W_0|$ jacobian accompanying the delta function imposing $DW_0 = 0$. Finally, the $\mathcal{F}_{ijk}$ are the "Yukawa couplings" characterizing the special geometry of the complex structure moduli space, i.e.

$$\mathcal{F}_{ijk} = e^{\mathcal{K}_c} \int_Z \Omega \wedge \partial_i \partial_j \partial_k \Omega \,. \tag{A.3}$$

Since we are interested in very small values of the cosmological constant, i.e. $\lambda_* \ll 1$, the $\mathcal{O}(X)$ part can be dropped from the integrand and the integral over X simply gives a factor $\pi\lambda_*$.

By rewriting the determinant in the integrand as a gaussian fermionic integral and the bosonic integral as a Laplace transformed gaussian [8], and then doing the bosonic integral, this expression can be rewritten as a fermionic integral with quartic fermionic action, hence it reduces to a finite number of terms. (Alternatively, one could stick to bosonic variables and apply Wick's theorem.) This is useful for low n, but for large n the number of terms becomes enormous, about n^{4n}, and straightforward numerical evaluation becomes impossible. Instead, we want to rewrite (A.2) in a form suitable for Monte Carlo estimates.

To this end, we implement the constraint $\|Z\|^2 \leq L_*$ by inserting $\int_0^{L_*} d\ell\, \delta(\ell - \|Z\|^2)$ in the integral. Changing variables from Z to U with $Z^i = \sqrt{\ell}\, U^i$, and doing the ℓ-integral then gives

$$\nu(z) = \frac{2^{2n+2}\pi\lambda_* L_*^{2n+1}}{(2n+1)\det g} \int d^{2n}U\, \delta(\|U\|^2 - 1) \left|\det \begin{pmatrix} 0 & U_j \\ U_i & \mathcal{F}_{ijk}\bar{U}^k \end{pmatrix}\right|^2 . \tag{A.4}$$

Now define the following "spherical" average, for any function $f(U)$:

$$\langle f(U)\rangle_{\|U\|=1} = \frac{\int d^{2n}U\, \delta(\|U\|^2 - 1)\, f(U)}{\int d^{2n}U\, \delta(\|U\|^2 - 1).} \tag{A.5}$$

Noting that

$$\int d^{2n}U\, \delta(\|U\|^2 - 1) = \frac{\pi^n}{(n-1)!\det g} \tag{A.6}$$

we can thus write

$$\nu(z) = \frac{2^{2n+2}\pi^{n+1}\lambda_* L_*^{2n+1}}{(2n+1)(n-1)!}\, \mu(z) \tag{A.7}$$

with

$$\mu(z) = \frac{1}{(\det g)^2} \left\langle \left|\det \begin{pmatrix} 0 & U_j \\ U_i & \mathcal{F}_{ijk}\bar{U}^k \end{pmatrix}\right|^2 \right\rangle_{\|U\|=1} . \tag{A.8}$$

Comparing this to (4.1), we get

$$\rho_0(z) = \frac{(2n+2)(2n)!}{\pi^{n+1}(n-1)!}\, \mu(z)\,. \tag{A.9}$$

This can be made more explicit in the large complex structure limit of Z, parametrized with the special coordinates $t^i = x^i + i\, y^i$, where y^i becomes large and $x_i \in [0,1]$. All data is encoded in the triple intersection numbers $\tilde{D}_{ijk}$ of the mirror $\tilde{Z}$ to Z. The Kähler potential is $\mathcal{K}_c = -\ln(2^3\tilde{V})$, where $\tilde{V} = \frac{1}{6}\tilde{D}_{ijk}y^i y^j y^k$. The metric and its determinant are

$$g_{i\bar{j}} = \frac{\tilde{V}_i\tilde{V}_j}{4\tilde{V}^2} - \frac{\tilde{V}_{ij}}{4\tilde{V}}\,, \qquad \det g = \frac{(-1)^{n+1}\det \tilde{V}_{ij}}{2^{2n+1}\tilde{V}^n}\,, \tag{A.10}$$

where $\tilde{V}_i = \partial_i\tilde{V} = \frac{1}{2}\tilde{D}_{ijk}y^j y^k$ and $\tilde{V}_{ij} = \partial_i\partial_j\tilde{V} = \tilde{D}_{ijk}y^k$. Finally, the Yukawa couplings are $\mathcal{F}_{ijk} = e^{\mathcal{K}_c} \tilde{D}_{ijk}$. Using all this and pulling the factor $e^{\mathcal{K}_c}$ out of the determinant, (A.8) becomes

$$\mu(y) = \frac{1}{4^{n-4}} \frac{\tilde{V}^2}{(\det \tilde{V}_{ij})^2} \left\langle \left| \det \begin{pmatrix} 0 & U_j \\ U_i & \tilde{D}_{ijk} \bar{U}^k \end{pmatrix} \right|^2 \right\rangle_{\|U\|=1} \tag{A.11}$$

Note that $\mu(y)$ is invariant under rescaling $y^i \to \lambda y^i$.

The average $\langle f(U) \rangle$ can be estimated numerically using Monte Carlo methods; for many variables this is in fact the only possible way. In the simplest version, one repeatedly picks a random vector Z from a normal distribution with mean zero and covariance matrix proportional to the metric $g_{i\bar{j}}$ (or any other distribution depending only on $\|Z\|$), one evaluates $f(U)$ with $U \equiv Z/\|Z\|$, and at the end one computes the average of the values obtained. This gives an approximate value for $\rho_0(y)$, the approximation becoming better with increasing number of sampling points.

What remains to be computed then to get the number of flux vacua is the integral (4.1). Again this can be done by Monte Carlo integration. However, computing $\rho_0(y)$ at every sample point in the way described above would require far too much computation time in the models we consider. Therefore we will replace $\rho_0(y)$ by its average over a limited number of sample points. This is a reasonable thing to do, since in examples $\rho_0(y)$ stays of more or less the same order of magnitude over the large complex structure part of moduli space (in particular, as noted before, it is scale invariant). The integral then becomes proportional to the volume of the part of moduli space under consideration, which can estimated numerically in reasonable time.

One important universal property of this integral can be deduced directly: vacua are strongly suppressed at large complex structure. To see this, note that under rescaling $y \to \lambda y$, $d^n y \det g \to \lambda^{-n} d^n y \det g$, so if for instance the region under consideration is given by $\tilde{V} > V_*$, then

$$N_{\rm vac}(\tilde{V} > V_*) \propto \frac{1}{V_*^{n/3}} \,. \tag{A.12}$$

We discuss this in more detail in [75].

B. Taming the complex structure moduli space of the $\mathcal{F}_{18}$ model

The Calabi-Yau 3-fold Z of the $\mathcal{F}_{18}$ model of section 5.1 has 89 complex structure moduli. Describing this space completely including its exact periods would be extremely complex, so we restrict ourselves to the large complex structure limit, which can be constructed as the classical Kähler moduli space of the mirror $\tilde{Z}$. Even this poses quite a challenge.

The classical Kähler moduli space of $\tilde{Z}$ is specified entirely in terms of the triple intersection numbers of the divisors $\tilde{D}_i$. These in turn can be computed from the quadruple intersection numbers of the divisors of the ambient toric variety $\tilde{Y}$. There are more than 100 independent divisors in $\tilde{Y}$, so a priori there are more than $100^4 = 10^8$ intersection numbers to compute.

Before one can start doing this, one needs a maximal triangulation of the fan of the toric variety. Although the fan in this case is very large (116 points), we managed to do this by hand, and found 576 cones of volume 1.

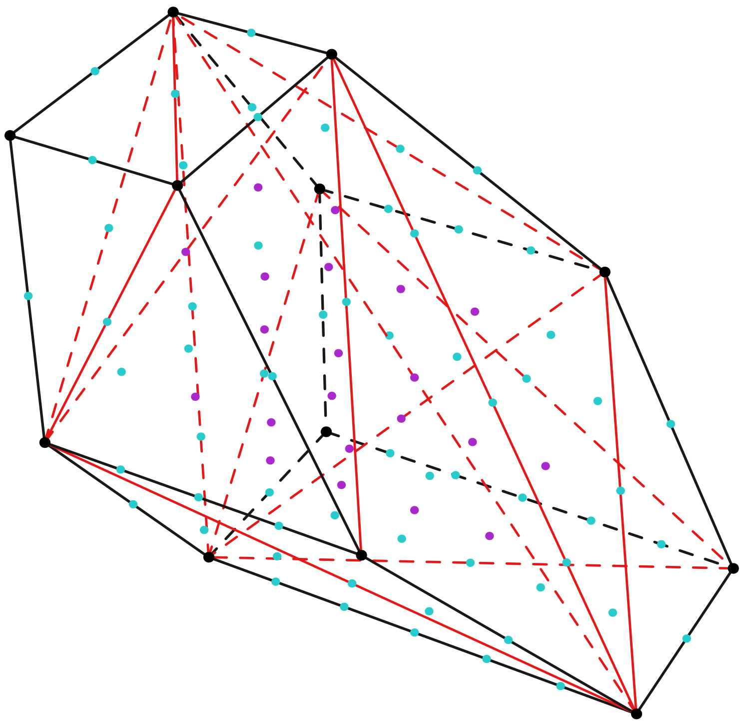

Figure 1: Partial triangulation of a 3-face of the polyhedron. Black dots correspond to vertices, blue dots correspond to points lying on edges or codimension two faces, while purple dots correspond to points interior to the 3-face.

The next step is to determine the actual Kähler cone within the $\mathbb{R}^{89}$ parameter space, i.e. the cone in which all holomorphic curves have positive area. This involves constructing and solving 377 inequalities in 89 variables, which is more difficult than one might expect. Even finding just one point satisfying these inequalities takes several hours on a PC.

In the end we want to compute numerically the volume of a subspace of the Kähler cone, as described in appendix A. Because of the high dimensionality of the space, this needs to be done by Monte Carlo. The integrand is relatively costly to evaluate, as it involves computing the 89×89 determinant given in (A.10). On a 2.4 GHz Pentium, evaluation using 10^7 Monte Carlo sample points took about three days.

In the following we outline how we proceeded to achieve these goals.

B.1 Intersections

The toric variety $\tilde{Y}$ has 116 toric divisors $\tilde{D}_i$, which correspond to points $p^i \in \mathbb{R}^4$ in the polyhedron $\nabla_{\tilde{Y}}$. The vertices of $\nabla_{\tilde{Y}}$ are given by

$$
\begin{array}{lllll}
(4,0,-2,1)\,, & (-2,-2,0,1)\,, & (-2,2,0,1)\,, & (0,0,0,-1)\,, & (2,2,0,1)\,, \\
(-2,0,2,1)\,, & (0,0,2,1)\,, & (-2,2,-2,1)\,, & (4,2,-2,1)\,, & (2,-2,0,1)\,, \\
(-2,-2,2,1)\,, & (0,-2,2,1)\,, & (-2,0,-2,1)\,. & &
\end{array}
$$

The triangulation of the polytope has 576 cones $\tilde{D}_i \tilde{D}_j \tilde{D}_k \tilde{D}_l$, all of volume 1.

The rule for the intersection product of four *distinct* divisors is the following: if the four divisors span a cone, their intersection is 1, otherwise it is 0. It is more complicated to find intersections where some divisors are the same, such as $\tilde{D}_i \cdot \tilde{D}_i \cdot \tilde{D}_j \cdot \tilde{D}_k$. This is done by making use of the 4 linear equivalence relations that exist between the 116 toric divisors, which can be simply read off from the points: $p^i_\mu \tilde{D}_i = 0$. This gives four equations $p^l_\mu \tilde{D}_{lijk} = 0$ for each distinct triple ijk. The unknowns are the double index intersection numbers $\tilde{D}_{iijk}$, so there are (more than) enough equations to find these. Solve this using a computer as one system of millions of equations in millions of variables would take a lot of time and memory. However, the problem can be split up in a much smaller number of systems of four equations in three variables: for a given $i < j < k$, the three variables are $\tilde{D}_{iijk}$, $\tilde{D}_{ijjk}$, $\tilde{D}_{ijkk}$, and the only ijk which need to be taken into consideration are those for which $\tilde{D}_i \tilde{D}_j \tilde{D}_k$ actually appears somewhere as a face in the list of cones. For other ijk the three unknowns are trivially zero. A similar reasoning can be followed to compute intersections with three and four identical indices. Thus almost all intersections vanish, and the remaining ones can be computed by computer in less than a minute.

Let us now turn to the Calabi-Yau hypersurface $\tilde{Z}$ in $\tilde{Y}$. Of the 116 divisors $\tilde{D}_i$ in $\tilde{Y}$, only 93 intersect $\tilde{Z}$ and descend to divisors on the Calabi-Yau (89 of those are independent). We will denote these divisors by the same $\tilde{D}_i$. Their triple intersection numbers are now easy to compute. They are simply given by the intersection with the anticanonical divisor: $\tilde{D}_{ijk} = \sum_l \tilde{D}_{ijkl}$. The resulting volume function is $\tilde{V} = \frac{1}{6} \tilde{D}_{ijk} x^i x^j x^k$.

Finally, the large complex structure prepotential of the original Calabi-Yau Z is given by this expression with the x^i replaced by complex variables, and takes the form

$$\mathcal{F} = {t_1}^3 - {t_2}^2 t_3 + 780 \text{ more terms.}$$

It is available upon request.

B.2 Mori cone

A class in $H_2(\tilde{Z})$ is specified by its intersections with the divisors $\tilde{D}_i$. It is important to know which of these classes can effectively be realized as holomorphic curves. The set of such effective curves forms a cone, called the Mori cone. The generators of the Mori cone C_a, given by their intersection numbers C_{ai} with the divisors $\tilde{D}_i$, are contained in the list of 'special' linear relations $C_{ai} p^i = 0$ between the polyhedron points p^i corresponding to the divisors. There is one such special relation for each adjacent pair of cones in the triangulation of the polyhedron, found as follows. Denote the three common points of the two cones by f_1, f_2 and f_3, and the two additional points by p and q. Then there will be a relation $p + q + n_1 f_1 + n_2 f_2 + n_3 f_3 = 0$, with the n_i integer.

Applying this to our model gives 579 relations. These correspond to effective curves of $\tilde{Y}$. What we want however are effective curves of $\tilde{Z}$, and these are obtained by keeping only the curves having zero intersections with the 23 divisors of $\tilde{Y}$ which do not descend to divisors of $\tilde{Z}$. This reduces the list to 377. Only a subset of this list will constitute a basis of generators of the Mori cone: any curve that is a positive linear combination of the others can be dropped. A basis can thus be constructed in steps as follows. Start with the first curve. Add to this the second one and check if in the resulting set any one of the curves is a positive multiple of the other. If so, remove this curve. Then add the third curve and remove any curve in the resulting set that is a positive linear combination of the others. Then add the fourth curve, and so on, till all 377 candidates have been considered. This procedure is much faster than starting with all curves and removing the dependent ones one by one, because verifying if a given vector equals some *positive* linear combination of a set of n vectors becomes nontrivial if n is bigger than the dimension, and with n substantially bigger, it is computationally extremely expensive (hours for $n = 377$ in this case).

The resulting basis of the Mori cone consists of 111 generators C_a.

B.3 Kähler cone

The dual of the Mori cone is the Kähler cone, that is the set of all $J = x^i D_i$ such that $C_a \cdot J > 0$. Note that this is a tiny fraction of the $\mathbb{R}^{89}$ parameter space; an estimate for the probability of a random point to be in the Kähler cone is $(1/2)^{111} \sim 10^{-34}$. Since we want to integrate over the Kähler cone, it is important to have a good parametrization of it — multiplying the integrand by a step function with support on the Kähler cone and Monte Carlo integrating over all x certainly won't do the job, since effectively all sample points will integrate to zero. Ideally, one would construct the exact generators K_p of the Kähler cone and write $J = t^p K_p$, $t^p > 0$. There exists an algorithm to find these generators, decribed in [84] p.11. Unfortunately, because the Mori cone is far from simplicial, this involves running over $\binom{111}{88} \sim 10^{23}$ candidate generators (the rays obtained by intersecting 88 of the 111 zero planes). It would take about the age of the universe to complete this task on a PC. Nevertheless, with some luck, we were able to construct an approximate parametrization. Applying Mathematica's function `InequalityInstance` to find one solution to our set of 111 inequalities results in a point which has exactly 89 curve areas $C_i \cdot J$ equal to 1, and 22 bigger than 1. Taking the areas of these special curves as coordinates y_i, it turns out that for uniformly random positive y values, the resulting J generically lies inside the Kahler cone. The integral over the Kähler cone can therefore be done by Monte Carlo integrating over all positive y and multiplying the integrand by a step function with support on the points satisfying the remaining 22 inequalities. In this way, not too many sample points evaluate to zero.

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